Lectures on differential topology

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Abstract

These are collected lecture notes on differential topology.

Contents

Introduction

These are the collected lecture notes on differential topology. They are based on [BJ82, GP10, BT82, Wal16]. Our reference for multivariable calculus is [DK04a, DK04b].

Differential topology is the study of smooth manifolds; topological spaces on which one can make sense of smooth functions. This is done by providing local coordinates. Through these, many of the results of multivariable calculus can be extended to manifolds. The latter provide a convenient language, the former the technical details: *state globally, prove locally*.

The motivating goal of differential topology is the classification of smooth manifolds, and maps between smooth manifolds. This is done through numerical invariants extracted from geometric objects living in our manifolds (e.g. submanifolds) or on our manifolds (e.g. differential forms). Particular instances of these ideas are *intersection theory* and *de Rham cohomology*.

Acknowledgments

Thanks to the students of Math132 and MAT1300HF for many corrections.

Chapter 1

Spheres in Euclidean space

In this first lecture we give a taste of differential topology, with a discussion of spheres which are embedded or immersed in $\mathbb{R}^k$. The highlight will be Smale's result that the two-dimensional sphere can be everted. Along the way, we meet a significant portion of the cast of this course: smooth manifolds, embeddings, isotopies, orientations, immersions, regular homotopies, winding numbers, and transversality.

1.1 Circle eversion

We are all familiar with the circle

$$S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

which we can thicken to an open annulus

$$\mathbb{A}^2 := \{(x, y) \in \mathbb{R}^2 \mid (1 + \delta)^{-1} < x^2 + y^2 < 1 + \delta\}$$

for some small $\delta > 0$. There is of course a standard inclusion inc of $\mathbb{A}^2$ into $\mathbb{R}^2$, given by sending $(x, y) \in \mathbb{A}^2$ to $(x, y) \in \mathbb{R}^2$.

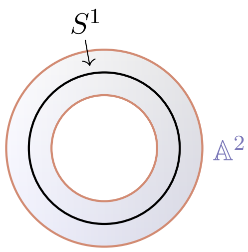


Figure 1.1 The circle S^1 inside the annulus $\mathbb{A}^2$.

There are many other inclusions of $\mathbb{A}^2$ into $\mathbb{R}^2$. We could rotate by 90° degrees counterclockwise

$$\begin{aligned} \mathrm{rot}_{90} \colon \mathbb{A}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-y, x), \end{aligned}$$

reflect in the x-axis

$$\begin{aligned} \mathrm{refl}\colon \mathbb{A}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x,-y), \end{aligned}$$

or invert the circle

$$\begin{aligned} \mathrm{inv}\colon \mathbb{A}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right), \end{aligned}$$

These injective maps are not only continuous, but have three further properties. Firstly, they are smooth: all partial derivatives exist and are continuous at each point in $(x,y) \in \mathbb{A}^2$. Secondly, not only does the total derivative exists at each point, but it is injective (in fact, invertible). Thirdly, they are homeomorphisms onto their image.

Definition 1.1.1. A continuous map $\mathbb{A}^2 \to \mathbb{R}^2$ is called an *embedding* if it is a smooth map which is a homeomorphism on its image and whose total derivative is injective everywhere.

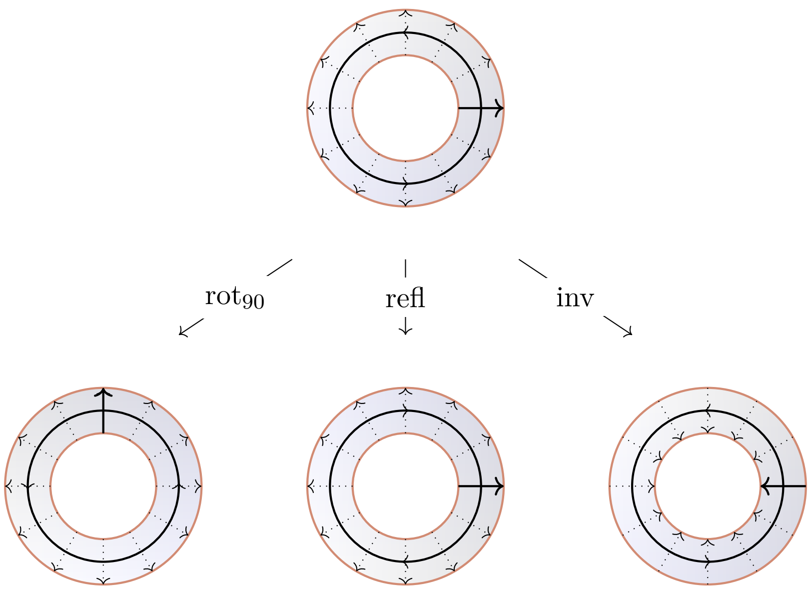


Figure 1.2 Three embeddings $\mathbb{A}^2 \hookrightarrow \mathbb{R}^2$.

How different are these embeddings from each other? The maps inc and rot$_{90}$ are closely related to each other: they can be connected by a path of embeddings. This path is given by varying the rotation angle

$$\begin{aligned} \mathrm{rot}_t\colon [0,1] \times \mathbb{A}^2 &\longrightarrow \mathbb{R}^2 \\ (t,(x,y)) &\longmapsto \left(\cos(\tfrac{\pi}{2}\cdot t)x + \sin(\tfrac{\pi}{2}\cdot t)y, -\sin(\tfrac{\pi}{2}\cdot t)x + \cos(\tfrac{\pi}{2}\cdot t)y\right), \end{aligned}$$

a path of embeddings. It is called an *isotopy* because it is also smooth as a map with domain $[0,1] \times \mathbb{A}^2$.

However, the cases of reflection and inversion are more subtle.

Proposition 1.1.2. *Both* refl *and* inv *can not be connected to* inc *(or equivalently* rot_{90}*) by such an isotopy.*

Proof. The reason is that both refl and inv reverse orientations. The Euclidean space $\mathbb{R}^2$ has a so-called *orientation*, given by a consistent choice of direction of "counterclockwise rotation," and so does $\mathbb{A}^2$ as an open subset of $\mathbb{R}^2$. As can be seen in Figure 1.3, rotations such as rot_{90} preserve orientation, but reflection refl and inversion inv do not.

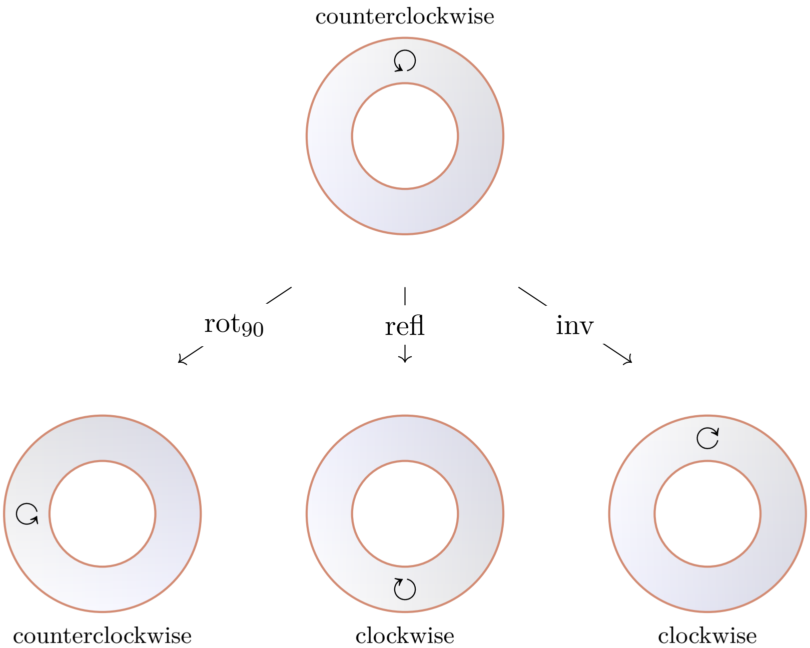


Figure 1.3 The effect of our three embeddings on orientations.

If the inclusion inc and reflection refl (or inversion inv) were isotopic then the latter would have to preserve orientation, because inc does and the embeddings in an isotopy can not switch from being orientation-preserving to being orientation-reversing. (This is the crux of the argument, and making it rigorous is something we will do in these notes.) $\square$

However, the composition of reflection and inversion does preserve orientation; reversing orientation twice preserves it. This map

$$\begin{aligned} \mathrm{eve} := \mathrm{inv} \circ \mathrm{refl} \colon \mathbb{A}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right) \end{aligned}$$

is called *eversion*. Can eversion be connected to the identity by an isotopy?

To answer this question, we look at $S^1 \subset \mathbb{A}^2$. This is our first example of a smooth manifold which is not Euclidean space $\mathbb{R}^n$ or an open subset thereof. More precisely, it is a *one-dimensional smooth manifold*; a topological space which locally looks like $\mathbb{R}$ and on which we can make sense of smooth functions. To do the latter, we use local coordinates on S^1 and our understanding of smooth maps between open subsets of Euclidean space: the two *charts* ("coordinate patches")

$$\begin{aligned} \phi_0\colon (0, 2\pi) &\longrightarrow S^1 \\ \theta &\longmapsto (\cos(\theta), \sin(\theta)) \\ \phi_1\colon (0, 2\pi) &\longrightarrow S^1 \\ \theta &\longmapsto (\cos(\theta + \pi), \sin(\theta + \pi)) \end{aligned}$$

cover all of S^1, and we say that $f\colon S^1 \to \mathbb{R}^2$ is *smooth* if both $f \circ \phi_0$ and $f \circ \phi_1$ are smooth. Similarly, it is an *embedding* if it is a smooth map which is a homeomorphism onto its image and whose total derivative is injective everywhere. It is easy to recognize it is a homeomorphism on its image; when we restrict the target to its image we get a continuous bijection between compact Hausdorff spaces.

If $\mathrm{eve}\colon \mathbb{A}^2 \to \mathbb{R}^2$ were isotopic to inc, then by restricting the isotopy to S^1 we would be able to prove that $\mathrm{eve}|_{S^1}$ is isotopic to $\mathrm{inc}|_{S^1}$. So, to prove that eve is not isotopic to inc, it suffices to show that $\mathrm{eve}|_{S^1}$ is not isotopic to $\mathrm{inc}|_{S^1}$.

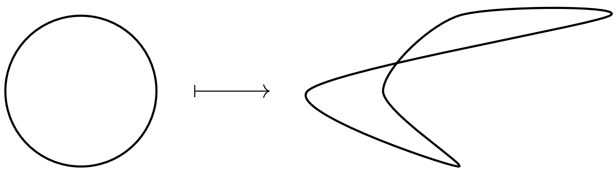


Figure 1.4 An example of the image of S^1 under an immersion into $\mathbb{R}^2$.

In fact, we will prove something even stronger. We can drop the condition that an embedding is injective. Since the derivative controls the local behaviour of smooth maps, that the derivative is everywhere non-zero means it is still locally injective. A smooth map $S^1 \to \mathbb{R}^2$ with everywhere non-zero derivative is called an *immersion*, and a smooth map $[0, 1] \times S^1 \to \mathbb{R}^2$ consisting of immersions is called a *regular homotopy*. This is a family of smooth maps where we allow self-intersections to occur, but not the pulling tight of loops (the derivative would blow up there).

Proposition 1.1.3. *The embeddings* $\mathrm{eve}|_{S^1}$ *and* $\mathrm{inc}|_{S^1}$ *are not regularly homotopic.*

Proof. Suppose a regular homotopy $e_t\colon [0, 1] \times S^1 \to \mathbb{R}^2$ existed between $\mathrm{eve}|_{S^1}$ and $\mathrm{inc}|_{S^1}$, then for each $s \in [0, 1]$, the map $e_s\colon S^1 \to \mathbb{R}^2$ is an immersion. Thus, when we take for $\theta_0 \in S^1$ the derivative $\frac{d}{d\theta}|_{\theta=\theta_0} e_s(\cos(\theta), \sin(\theta))$ we get a *non-zero* vector in $\mathbb{R}^2$. If we normalize these to have length 1, we get a smooth map

$$\mathrm{gauss}(e_s)\colon S^1 \longrightarrow S^1.$$

Here the domain S^1 is the circle which is the domain of our immersions, and the target S^1 is the space of unit length vectors in $\mathbb{R}^2$. You can think of the latter as the space of lines through the origin in $\mathbb{R}^2$ with a choice of orthonormal basis (in this case just a single vector).

If $\mathrm{eve}|_{S^1}$ and $\mathrm{inc}|_{S^1}$ are regularly homotopic through e_t, then $\mathrm{gauss}(\mathrm{eve}|_{S^1})$ and $\mathrm{gauss}(\mathrm{inc}|_{S^1})$ can be connected the path $\mathrm{gauss}(e_s)$ of maps $S^1 \to S^1$. In other words, they would be *homotopic*. But they are not; as $\mathrm{gauss}(\mathrm{eve}|_{S^1}) = \mathrm{refl} \circ \mathrm{rot}_{90}$ and $\mathrm{gauss}(\mathrm{inc}|_{S^1}) = \mathrm{rot}_{90}$ wind around the origin a different number of times; the first once clockwise (so -1 times) and the second once counterclockwise (so 1 times). The difference between these *winding numbers* implies that $\mathrm{gauss}(\mathrm{eve}|_{S^1})$ and $\mathrm{gauss}(\mathrm{inc}|_{S^1})$ are not homotopic. (Again, this is the crux and we need to rigorously justify this claim.) $\square$

1.2 Knots

Let us now increase the dimension of the target; instead of looking at circles in $\mathbb{R}^2$ we will look at circles in $\mathbb{R}^3$. Immersions are significantly easier to study than embeddings; though both are smooth maps with injective total derivative, a *local* condition, embeddings need to be injective, a *global* condition. This distinction becomes evident when we try to discern the difference between embeddings $S^1 \hookrightarrow \mathbb{R}^3$ and immersions $S^1 \looparrowright \mathbb{R}^3$.

Proposition 1.2.1. *Each immersion $S^1 \looparrowright \mathbb{R}^3$ is regularly homotopic to an embedding.*

Proof. This uses a technique called *transversality*. Informally, this allows you take smooth maps to be "generic" without loss of generality. This means that by making an arbitrary small change to an immersion $e_0 \colon S^1 \to \mathbb{R}^3$, we can make its self-intersections have the "expected dimension."

Here "arbitrarily small" means that for each $\epsilon > 0$, we can find an $e_1 \colon S^1 \to \mathbb{R}^3$ whose values and derivatives are within ϵ of those for e_0. By taking ϵ to be small enough, during a linear interpolation

$$\begin{aligned} e_t \colon S^1 \times [0,1] &\longrightarrow \mathbb{R}^3 \\ (t,\theta) &\longmapsto (1-t)\cdot e_0(\theta) + t\cdot e_1(\theta) \end{aligned}$$

the derivative never becomes 0. In particularly, e_0 is regularly homotopic to e_1.

The advantage of e_1 is that its self-intersections have the expected dimension. This expected dimension is that of the intersection of two affine lines $\mathbb{R}^3$ with arbitrarily chosen coefficients: two such lines do not intersect, and thus generically the self-intersections are empty as well. $\square$

Isotopy classes of embeddings $S^1 \hookrightarrow \mathbb{R}^3$ are called *knots*. The isotopy class of the standard circle $\mathrm{inc} \colon S^1 \hookrightarrow \mathbb{R}^3$ is the *unknot*, but there are of course many more interesting and complicated knots. At first sight many seem obviously distinct, or at least non-trivial. This is an artefact of our tendency to draw rather simple knots: it is by no means clear to me that Figure 1.5 is not the unknot. It should furthermore not be obvious how to prove that two knots are distinct, as you need to rule out the existence of some extremely complicated isotopy. To do so one uses knot invariants, with such disparate sources as algebraic topology, combinatorics, number theory, hyperbolic geometry, or quantum field theory [Ada04, Sos23]. We will discuss some of these later. At any rate, your intuition is correct:

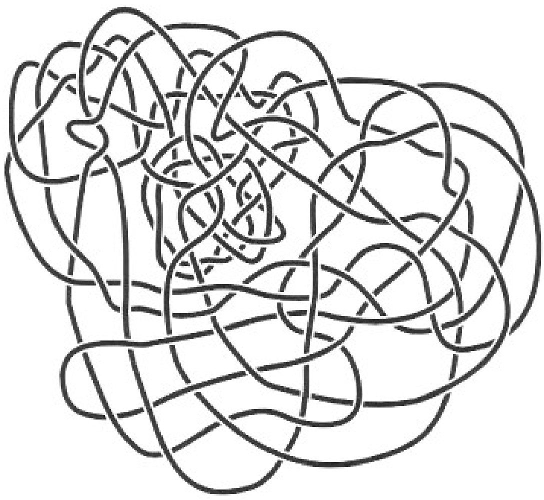


Figure 1.5 Haken's "gordian knot," which is actually unknotted (from [Sos23]).

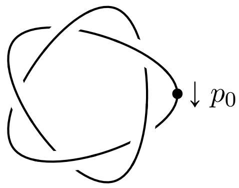
Proposition 1.2.2. *There are infinitely many distinct isotopy classes of embeddings $S^1 \hookrightarrow \mathbb{R}^3$. That is, there are infinitely many knots.*

Remark 1.2.3. This does not mean that distinguishing knots, or recognizing unknots, is easy. Even though there exists an algorithm that says whether a knot is the unknot, these algorithms are not very efficient [HLP99].

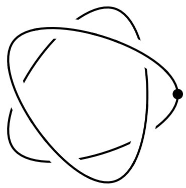
Armed with this knowledge, Proposition 1.2.1 seems rather useless. All we have shown is that immersions of a circle into $\mathbb{R}^3$ can be represented by knots. However, we can use that this representation is not unique. In particular, if we are interested in immersions we are allowed to make the strands of a knot self-intersect! Using this, it is not hard to give an informal proof of the following:

Proposition 1.2.4. *All immersions $S^1 \looparrowright \mathbb{R}^3$ are regularly homotopic.*

Proof sketch. By another application of transversality, it is possible to draw each knot as you are used to; a circle in the plane with some crossings, which never occur at the same point. As we just explained, you can change any crossing using a regular homotopy. Let us explain through an example a procedure to change crossings to end up with an unknot. Suppose our starting point is:



We fix a point p_0 in the knot, and start moving along it in an arbitrary direction. When we cross under a strand, we (a) keep it as it is if we haven't seen the crossing yet, but (ii) if we have seen it we change the crossing. For example, the first crossing clockwise from p_0 is not changed but the second one is. The result will be:



I will leave it to the reader to understand why this procedure always produces an unknot (hint: look at the height of the strands). □

1.3 Sphere eversion

Let us now increase the dimension of the domain. There is a two-dimensional sphere $S^2 := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$. This is a two-dimensional smooth manifold which is a subset of the thickened sphere $\mathbb{A}^3 := \{(x,y,z) \in \mathbb{R}^3 \mid (1+\delta)^{-1} < x^2+y^2+z^2 < 1+\delta\}$ for some small $\delta > 0$.

Again, in addition to the identity map $\mathrm{inc}\colon \mathbb{A}^3 \hookrightarrow \mathbb{R}^3$ there are many other inclusions; we could rotate by applying an element $A \in \mathrm{SO}(3)$ (the group of rotations around some axis through the origin in $\mathbb{R}^3$), reflect in the (x,y)-plane

$$\begin{aligned} \mathrm{refl}\colon \mathbb{A}^3 &\longrightarrow \mathbb{R}^3 \\ (x,y,z) &\longmapsto (x,y,-z), \end{aligned}$$

or invert it

$$\begin{aligned} \mathrm{inv}\colon \mathbb{A}^3 &\longrightarrow \mathbb{R}^3 \\ (x,y,z) &\longmapsto \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right). \end{aligned}$$

All of these are smooth maps, and in fact embeddings. The rotation by A is isotopic to the identity because the group $\mathrm{SO}(3)$ is path-connected (move the rotation angle to 0), while both refl and inv are not isotopic to the identity because they do not preserve the orientation.

However, the eversion

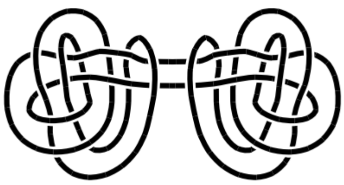
$$\begin{aligned} \mathrm{eve} := \mathrm{refl} \circ \mathrm{inv} \colon \mathbb{A}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{-z}{x^2 + y^2 + z^2} \right) \end{aligned}$$

does preserve the orientation. Is it isotopic to the identity? The answer turns out to be negative; in fact, $\mathrm{eve}|_{S^2}$ is already not isotopic to $\mathrm{inc}|_{S^2}$. If it were, we could "drag along" the disk $D^3 \subset \mathbb{R}^3$ that bounds the image of $\mathrm{inc}|_{S^2}$ on the inside along an isotopy of embeddings—a result called *isotopy extension* (which requires the embeddings and isotopy are proper)—and would have to end up with a disk that bounds the image of $\mathrm{eve}|_{S^2}$ *on the outside*, which is clearly impossible. (This requires justification.)

However, it is a surprising result of Smale that $\mathrm{eve}|_{S^2}$ *is* regularly homotopic to $\mathrm{inc}|_{S^2}$ [Sma58]. That is, these two embeddings can be connected by a family of immersions; self-intersections are allowed to form, but not the pulling tight of the fabric of S^2. The procedure is rather complicated, but you can watch a video of it called *Outside In* online. The reason this works is that the two-dimensional versions of the Gauss maps, $\mathrm{gauss}(\mathrm{eve}|_{S^2})$ and $\mathrm{gauss}(\mathrm{inc}|_{S^2})$, which are maps from S^2 to the space $V_2(\mathbb{R}^3)$ of two-dimensional planes though origin with a choice of orthonormal basis, are homotopic. This homotopy can then be approximated by a regular homotopy using *holonomic approximation*, a instance of general philosophy called an *h-principle* [EM02]. Explicitly implementing this approximation gives the video referred to above.

1.4 Problems

Problem 1. Is the following knot trivial (i.e. isotopic to the unknot)?



If no, explain why. If yes, draw an isotopy.

Chapter 2

Smooth manifolds

In this lecture we give the modern definition of a smooth manifold, which is the one we will use throughout this course; you have read what is considered to be the historically first one, due to Riemann. It is given in [BJ82, Chapter 1], but unfortunately not in [GP10]. References for further reading are [Tu11, Chapter 5] or [Wal16, Section 1.1]. We also give a number of examples (you need to know S^n, $\mathbb{R}P^n$, and $\mathbb{C}P^n$, but not the examples of moduli spaces).

2.1 Topological manifolds

Underlying every smooth manifold is a topological manifold. This is a topological space which locally looks like Euclidean space, though we will ask it satisfies some point-set topological conditions to make it more well-behaved. A local property of a topological space is one which concerns sufficiently small open subsets. For a k-dimensional topological manifold the relevant local condition is "being homeomorphic to an open subset of $\mathbb{R}^k$:"

Definition 2.1.1. A topological space X is *locally Euclidean of dimension k* if each point $x \in X$ has an open neighbourhood $V_x \subset X$ which is homeomorphic to an open subset $U_x \subset \mathbb{R}^k$.

This models a "world" which, for a tiny creature living in it, is indistinguishable from $\mathbb{R}^k$. This intuition is not compatible with certain pathological examples. The "world" is not supposed to "split into two points" somewhere, as occurs in a plane with doubled origin [SS95, §74]. This is ruled out by demanding X is Hausdorff (any two distinct points have distinct open neighbourhoods). Furthermore, the "world" should admit a notion of distance, i.e. a metric. For a Hausdorff locally Euclidean topological space, being metrizable is equivalent to being second-countable (admitting a countable basis for its topology) [Gau09], and hence we demand that X is also second-countable. An example of a locally Euclidean space which is Hausdorff but not second-countable is the long line, created by "concatenating" uncountably many real lines [SS95, §45].

Definition 2.1.2. A *k-dimensional topological manifold* is a second-countable Hausdorff space X which is locally Euclidean of dimension k.

This definition only involves *properties* of X. We can rephrase the property that it is locally Euclidean as *data* instead, which will be necessary to define smooth manifolds.

Definition 2.1.3. A triple $(U_\alpha, V_\alpha, \phi_\alpha)$ of open subsets $U_\alpha \subset \mathbb{R}^k$, $V_\alpha \subset X$, and a homeomorphism $\phi_\alpha \colon U_\alpha \to V_\alpha$ is called a *chart* or a *local parametrization.*

Definition 2.1.4. A collection of charts $(U_\alpha, V_\alpha, \phi_\alpha)$ such that $\bigcup_\alpha V_\alpha = X$ is a *k-dimensional atlas* for X.

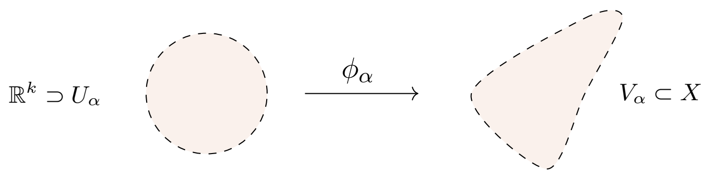


Figure 2.1 A chart.

Two local parametrizations $\phi_\alpha \colon \mathbb{R}^k \supset U_\alpha \to V_\alpha \subset X$ and $\phi_\beta \colon \mathbb{R}^k \supset U_\beta \to V_\beta \subset X$ give two competing identifications of $V_\alpha \cap V_\beta \subset X$ with an open subset of $\mathbb{R}^k$, which we can compare by the *transition function*

$$\psi_{\alpha\beta} := \phi_\beta^{-1} \circ \phi_\alpha \colon \mathbb{R}^k \supset \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \xrightarrow{\phi_\alpha} V_\alpha \cap V_\beta \xrightarrow{\phi_\beta^{-1}} \phi_\beta^{-1}(V_\alpha \cap V_\beta) \subset \mathbb{R}^k. \quad (2.1)$$

(It would be better to use the notation $\phi_\beta^{-1} \circ \phi_\alpha|_{\phi_\alpha^{-1}(V_\alpha \cap V_\beta)}$ to point out we are restricting the domain, but this notation would quickly become unwieldy.)

An atlas for a topological manifold X is not unique, but it turns out there is a unique maximal one. We shall not discuss this in detail now, saving a discussion of maximal atlases for smooth manifolds (where there is no longer a unique one, i.e. there are exotic smooth structures). An alternative equivalent definition of a k-dimensional topological manifold is then:

Definition 2.1.5. A *k-dimensional topological manifold* is a second-countable Hausdorff space X with a maximal k-dimensional atlas.

2.2 Smooth manifolds

On a topological manifold, as on any topological space X, we can make sense of continuous functions $X \to \mathbb{R}$. A smooth manifold is a refinement of a topological manifold with additional data that allows us to make sense of *smooth* functions $X \to \mathbb{R}$. This will use that we know from multivariable calculus what a smooth function $\mathbb{R}^k \to \mathbb{R}$ is: a map which has partial derivatives of arbitrary degree, in other words, an infinitely-many times differentiable function.

As the domain of a chart is an open subset of $\mathbb{R}^k$, we know what it means for a continuous function to be smooth with respect to the local coordinates provided by a chart. To make guarantee consistency between charts, we require that the transition functions $\psi_{\alpha\beta}$ are smooth.

Definition 2.2.1. A *k-dimensional smooth atlas* for a topological space X is a collection of triples $(U_\alpha, V_\alpha, \phi_\alpha)$ consisting of

- an open subset $U_\alpha \subset \mathbb{R}^k$,
- an open subset $V_\alpha \subset X$, and
- a homeomorphism $\phi_\alpha \colon U_\alpha \to V_\alpha$,

so that $\bigcup_\alpha V_\alpha = X$ and all maps

$$\psi_{\alpha\beta} = \phi_\beta^{-1} \circ \phi_\alpha \colon \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \longrightarrow \phi_\beta^{-1}(V_\alpha \cap V_\beta)$$

are smooth maps between open subsets of $\mathbb{R}^k$. The triples $(U_\alpha, V_\alpha, \phi_\alpha)$ are called *charts* and the maps $\phi_\beta^{-1} \circ \phi_\alpha$ are called *transition functions*.

Observe that these transition function have the following properties:

$$\psi_{\alpha\alpha} = \mathrm{id} \quad \text{and} \quad \psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}.$$

Taking $\gamma = \alpha$, this gives

$$\psi_{\alpha\beta} \circ \psi_{\beta\alpha} = \mathrm{id},$$

as smooth maps $\mathbb{R}^k \supset \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \to \phi_\beta^{-1}(V_\alpha \cap V_\beta) \subset \mathbb{R}^k$. This shows that $\psi_{\alpha\beta}$ is a smooth bijection with smooth inverse, and hence is what we call a *diffeomorphism*. Thus, in a smooth atlas the transition functions are always diffeomorphisms.

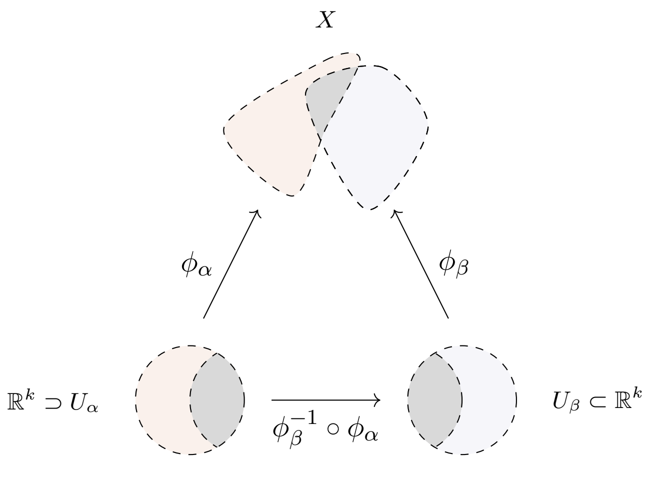


Figure 2.2 A transition function.

Two atlases for X are said to be *compatible* if their union is an atlas. A *maximal atlas* is one with the property that every atlas compatible with it, is in fact contained in it.

Lemma 2.2.2. *Every k-dimensional smooth atlas is contained in a unique maximal k-dimensional smooth atlas.*

Proof. For uniqueness, it suffices to prove that every two k-dimensional smooth atlases $\mathcal{A}' = \{(U'_\beta, V'_\beta, \phi'_\beta)\}$ and $\mathcal{A}'' = \{U''_\gamma, V''_\gamma, \phi''_\gamma)\}$ containing a given one $\mathcal{A} = \{(U_\alpha, V_\alpha, \phi_\alpha)\}$ are compatible. That is, we must prove that every map

$$(\phi''_\gamma)^{-1} \circ \phi'_\beta \colon (\phi'_\beta)^{-1}(V'_\beta \cap V''_\gamma) \to (\phi''_\gamma)^{-1}(V'_\beta \cap V''_\gamma)$$

is smooth. Since being smooth is a local property, it is enough to prove that each $x \in (\phi'_\beta)^{-1}(V'_\beta \cap V''_\gamma)$ has an open neighbourhood such that the restriction of $(\phi''_\gamma)^{-1} \circ \phi'_\beta$ to this open neighbourhood is smooth. Let us pick a chart $(U_\alpha, V_\alpha, \phi_\alpha) \in \mathcal{A}$ so that $\phi'_\beta(x) \in V_\alpha$. Then we can write the restriction of $(\phi''_\gamma)^{-1} \circ \phi'_\beta$ to $(\phi'_\beta)^{-1}(V_\alpha \cap V'_\beta \cap V''_\gamma)$ as

$$((\phi''_\gamma)^{-1} \circ \phi_\alpha) \circ (\phi_\alpha^{-1} \circ \phi'_\beta),$$

which is a composition of two smooth functions because both $\mathcal{A}'$ and $\mathcal{A}''$ are compatible with $\mathcal{A}$. Hence it is smooth, and hence so is $(\phi''_\gamma)^{-1} \circ \phi'_\beta$. Thus $\mathcal{A}'$ and $\mathcal{A}''$ are compatible.

Now that we have proven that $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{A} \subset \mathcal{A}''$ implies that $\mathcal{A}'$ and $\mathcal{A}''$ are compatible, we can just define

$$\mathcal{A}_{\max} := \bigcup_{\mathcal{A} \subset \mathcal{A}'} \mathcal{A}'. \qquad \square$$

Definition 2.2.3. A *k-dimensional smooth manifold* is a Hausdorff second-countable topological space X with a maximal k-dimensional smooth atlas.

That is, it is a k-dimensional topological manifold with an atlas where all transition functions are smooth. Some questions and answers about this definition:

(a) *How should I think of the smooth atlas?* The interpretation that follows directly from the definition is that it provides local coordinates, via the maps ϕ_α^{-1}, so that the transition between two of these coordinate systems is smooth. A different perspective on the role of an atlas is the one we used to motivate it: it tells you when a continuous function $f \colon X \to \mathbb{R}$ is *smooth*:

Definition 2.2.4. A continuous function $f \colon X \to \mathbb{R}$ is *smooth* when

$$f \circ \phi_\alpha \colon \mathbb{R}^k \supset U_\alpha \longrightarrow \mathbb{R}$$

is smooth for all charts $(U_\alpha, V_\alpha, \phi_\alpha)$.

This definition generalizes with ease to the case where the target is $\mathbb{R}^m$. To generalize to the case that the target is another smooth manifold, we involve the charts of the target. We discuss the following definition in more detail in the next lecture:

Definition 2.2.5. Let X and Y be manifolds with atlases $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ and $\{(U'_{\alpha'}, V'_{\alpha'}, \phi'_{\alpha'})\}$ respectively. A continuous map $f \colon X \to Y$ is *smooth* if

$$(\phi'_{\alpha'})^{-1} \circ f \circ \phi_\alpha \colon \mathbb{R}^k \supset \phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_{\alpha'})) \longrightarrow U'_{\alpha'} \subset \mathbb{R}^{k'}$$

is smooth for all charts.

(b) *When are two manifolds "the same"?* Saying when two manifolds are equivalent involves Definition 2.2.5:

Definition 2.2.6. A smooth map $f\colon X \to Y$ is a *diffeomorphism* if it is a bijection with smooth inverse.

Two manifolds X and Y are to be considered equivalent when there is a diffeomorphism between them; we say they are *diffeomorphic*.

(c) *Why demand X is Hausdorff and second-countable?* We mentioned that this "fits our intuitions," but there are more utilitarian answers. First, atlases on topological spaces without these properties rarely arise in practice. Second, having these properties is helpful, as they imply the existence of certain smooth functions $X \to \mathbb{R}$. For example, Hausdorffness will mean we can construct a smooth function which separates two distinct points, and both properties are used to construct partitions of unity.

(d) *Why demand that the atlas is maximal?* If we did not, then S^2 with two charts would be a different smooth manifold than S^2 with three charts. This would be absurd. Furthermore, we often want certain nice charts to exist. If our atlas has few charts this may not be the case. However, in practice we will want to specify a smooth manifold with an atlas that is as small as possible; a finite amount of data is easier to comprehend that an infinite amount. Then Lemma 2.2.2 generates for us a unique maximal atlas.

(e) *Can a topological space X have more than one maximal atlas?* The answer is almost always yes, as you can change the charts by a homeomorphism $X \to X$; the resulting smooth manifold is a diffeomorphic but the maximal atlases are not the same. On the homework you work this out. Moreover, even up to diffeomorphism a topological space X can have more than one maximal atlas. Another term for a maximal atlas is a *smooth structure*. Milnor surprised the mathematical community when he proved that S^7 admits more than one smooth structure up to diffeomorphism [Mil56a]; there are in fact 15.[1] This is a global phenomenon except when $n = 4$, as $\mathbb{R}^n$ admits a unique smooth structure up to diffeomorphism when $n \neq 4$. On the other hand, $\mathbb{R}^4$ admits uncountable many smooth structures up to diffeomorphism [Sco05, Section 5.4], and yes, you should be surprised by that.[2]

(f) *Can a topological space X have atlases of different dimensions?* This is not possible by a famous result of algebraic topology due to Brouwer called *invariance of domain*, which says that any injective map from an open subset of $\mathbb{R}^k$ to $\mathbb{R}^k$ has image given by an open subset [Hat02, Theorem 2.B.3]. If two such atlases did exist, charts from them would give a homeomorphism $f\colon \mathbb{R}^k \supset U \to V \subset \mathbb{R}^{k'}$ between open subsets of $\mathbb{R}^k$ and $\mathbb{R}^{k'}$ for say $k > k'$.

[1] The more well-known figure is that the group Θ_7 of oriented exotic spheres up to orientation-preserving homeomorphism is isomorphic to $\mathbb{Z}/28$. In this group, inverse is given by reversing the orientation, so that when we allow (not necessarily orientation-preserving) diffeomorphisms there are 15 elements, corresponding $\{0\}$, $\{14\}$ and $\{a, 28-a\}$ for $1 \leqslant a \leqslant 13$.

[2] It is more accurate to think of this as there being many distinct 4-dimensional smooth manifolds that for a magical reason happen to be homeomorphic to $\mathbb{R}^4$.

But invariance of domain implies that the composition of f with the inclusion

$$i \circ f \colon \mathbb{R}^k \supset U \longrightarrow V \subset \mathbb{R}^{k'} \longrightarrow \mathbb{R}^k$$

has image both an open subset of $\mathbb{R}^k$ and contained in the subset $\mathbb{R}^{k'} \subset \mathbb{R}^k$, which is impossible.

2.3 Examples of manifolds

2.3.1 First examples

Example 2.3.1 (Euclidean spaces). The prototypical example of a k-dimensional smooth manifold is $\mathbb{R}^k$ itself. It has second-countable and Hausdorff, and has an atlas with a single chart: $(U, V, \phi) = (\mathbb{R}^k, \mathbb{R}^k, \mathrm{id})$.

Example 2.3.2 (Spheres). Recall that the k-sphere is the subspace of $\mathbb{R}^{k+1}$ defined by

$$S^k := \left\{ (x_0, \ldots, x_k) \,\middle|\, \sum_{i=0}^{k} x_i^2 = 1 \right\}.$$

As a subspace of a second-countable Hausdorff topological space, it is second-countable and Hausdorff, Problem 2. We will now describe a k-dimensional smooth atlas on it, making it a k-dimensional smooth manifold, in terms of $2(k+1)$ different hemispheres. It suffices to describe the ϕ^{-1}'s (and then we can of course recover the ϕ's as their inverse). For $0 \leqslant j \leqslant k$ and $i \in \{0,1\}$, we have a chart given by

$$\begin{aligned} \phi_{ij}^{-1} \colon S^k \supset V_{ij} = \{x \in S^k \mid (-1)^i x_j > 0\} &\longrightarrow U_{ij} = \mathrm{int}(D^k) \subset \mathbb{R}^k \\ (x_0, \ldots, x_k) &\longmapsto (x_0, \cdots, \widehat{x_j}, \cdots, x_k). \end{aligned}$$

The transition functions have most entries of the form x_i, except that one has the form $\sqrt{1 - \sum_{i \neq j} x_i^2}$. These are clearly smooth.

A different but compatible k-dimensional smooth atlas with only two charts is given by stereographic projection. As before, we describe the ϕ^{-1}'s: if $C_N, C_S \subset S^k$ denote small closed neighborhoods of the north and south pole $(\pm 1, 0, \ldots, 0)$, then ϕ^{-1} is given by casting rays form N through $S^k \backslash C_N$ onto a plane below the sphere, see Figure 2.3.

Example 2.3.3 (Real projective spaces). The real projective space $\mathbb{R}P^k$ is the space of lines through the origin in $\mathbb{R}^{k+1}$. Such a line is specified by a unit vector, up to multiplication by ± 1. That is, it is the quotient space

$$\mathbb{R}P^k = S^k/\sim$$

with $\sim$ the equivalence relation generated by $(x_0, \ldots, x_k) \sim (-x_0, \ldots, -x_k)$. We will denote an example class as $[x_0 : \cdots : x_k]$.

The first of the atlases for S^n given in the previous example induces a k-dimensional smooth atlas on $\mathbb{R}P^k$. It has $(k+1)$ charts given as follows: for

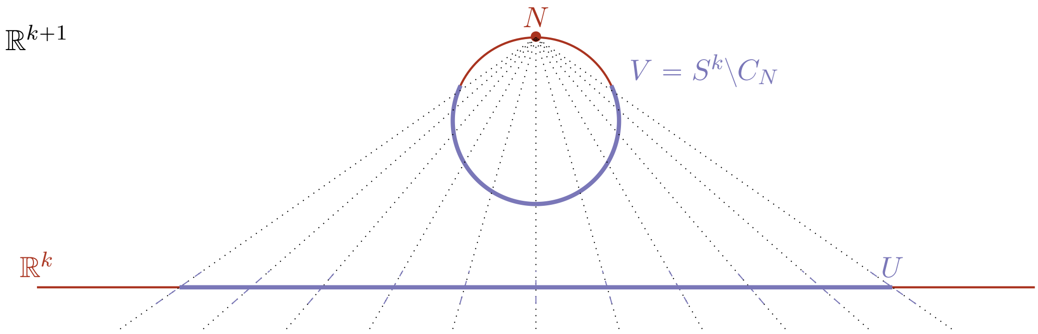


Figure 2.3 To obtain $\phi^{-1}: V \to U$, the inverse of a local parametrization of $S^k \subset \mathbb{R}^{k+1}$, follow the rays.

$0 \leqslant j \leqslant k$ it is

$$\begin{aligned}\overline{\phi}_j^{-1}: \mathbb{R}P^k \supset V_j = \{x \in \mathbb{R}P^k \mid x_j \neq 0\} &\longrightarrow U_j = \mathrm{int}(D^k) \subset \mathbb{R}^k \\ [x_0 : \ldots : x_k] &\longmapsto \mathrm{sign}(x_j)(x_0, \cdots, \hat{x}_j, \cdots, x_k).\end{aligned}$$

Example 2.3.4 (Surfaces of genus g). We will not describe atlases for these yet, but for each $g \geqslant 0$ there is a compact surface of genus g. It looks like a sphere with g handles added to it:

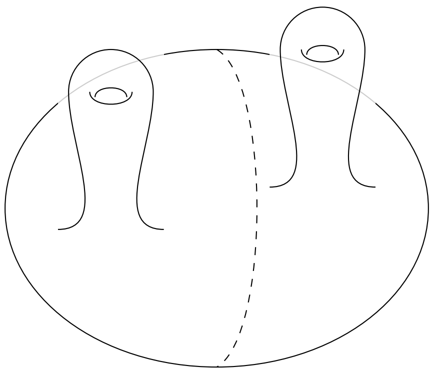


Figure 2.4 A surface of genus $g = 2$.

The classification of surfaces say that all compact orientable two-dimensional smooth manifolds (we will define "orientable manifolds" later) are diffeomorphic to Σ_g for some g.

2.3.2 Constructions of further manifolds

Example 2.3.5 (Open subsets). Suppose $U \subset X$ is an open subset of a k-dimensional smooth manifold. If $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ is an atlas of X, then the maps

$$\phi_\alpha|_{\phi_\alpha^{-1}(V_\alpha \cap U)}: \mathbb{R}^k \supset U_\alpha \supset \phi_\alpha^{-1}(V_\alpha \cap U) \longrightarrow V_\alpha \cap U \subset U$$

endow U with a k-dimensional smooth atlas. If the atlas of X is maximal, so is this atlas of U.

Example 2.3.6 (Disjoint unions). Let M and N be smooth manifolds with smooth atlases $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ and $\{(U'_\beta, V'_\beta, \phi'_\beta)\}$, of same dimension $m = n$. Then their union is an atlas for the disjoint union $M \sqcup N$, though it is in general not maximal even if the atlases on M and N are. This is the *disjoint union* of smooth manifolds.

Example 2.3.7 (Products). Now let M and N be smooth manifolds with smooth atlases $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ and $\{(U'_\beta, V'_\beta, \phi'_\beta)\}$, of dimension m and n respectively. Then the maps

$$\phi_\alpha \times \phi'_\beta \colon \mathbb{R}^m \times \mathbb{R}^n \supset U_\alpha \times U'_\beta \longrightarrow V_\alpha \times V'_\beta \subset M \times N$$

endow the cartesian product $M \times N$ with an $(m+n)$-dimensional smooth atlas, though it is in general not maximal even if the atlases of M and N are. This is the *product* of smooth manifolds.

Example 2.3.8 (Pre-manifolds). A *k-dimensional smooth pre-manifold* is a set X together with a collection $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ of $U_\alpha \subset \mathbb{R}^k$ an open subset, $V_\alpha \subset X$ a subset, and $\phi_\alpha \colon U_\alpha \to V_\alpha$ a bijection. We require that all maps

$$\psi_{\alpha\beta} = \phi_\beta^{-1} \circ \phi_\alpha \colon \mathbb{R}^k \supset \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \longrightarrow \phi_\beta^{-1}(V_\alpha \cap V_\beta) \subset \mathbb{R}^k$$

are smooth.

Then we can give X the smallest topology such that all ϕ_α are continuous. If this is Hausdorff and second countable, then $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ serves as a k-dimensional smooth atlas on X and hence makes it into a k-dimensional smooth manifold. In terms of category theory, this in fact presents X as a colimit of open subsets in Euclidean space.

2.3.3 Riemann's vision

In this more advanced section, we recall some historical context. You should not be surprised if much of this material is unfamiliar to you.

One-dimensional complex manifolds

If you have studied complex analysis, the following example may illuminate the definition of a k-dimensional smooth manifold.

We will define complex manifolds by replacing $\mathbb{R}$ by $\mathbb{C}$ and smooth maps by holomorphic maps: a 1-*dimensional complex atlas* for topological space X is a collection of triples $(U_\alpha, V_\alpha, \phi_\alpha)$ of an open subset $U_\alpha \subset \mathbb{C}$, an open subset $V_\alpha \subset X$, and a homeomorphism $\phi_\alpha \colon U_\alpha \to V_\alpha$, so that $\bigcup V_\alpha = X$ and all maps

$$\phi_\beta^{-1} \circ \phi_\alpha \colon \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \longrightarrow \phi_\beta^{-1}(V_\alpha \cap V_\beta)$$

are holomorphic maps between open subsets of $\mathbb{C}$. A 1-*dimensional complex manifold* is then a second-countable Hausdorff topological X with a maximal 1-dimensional complex atlas.

Since $\mathbb{C}$ can be identified with $\mathbb{R}^2$ and all holomorphic maps are smooth, any 1-dimensional complex manifold is a 2-dimensional smooth manifold. However, since it is much harder for a function to be holomorphic than for it to be smooth, it is harder to produce 1-dimensional complex manifolds than 2-dimensional smooth manifold.

Remark 2.3.9. By replacing $\mathbb{C}$ by $\mathbb{C}^k$, this definition generalizes to that of a k-dimensional complex manifold. Such a complex manifold always gives rise to a $2k$-dimensional smooth manifold.

The moduli spaces of Riemann surfaces

It is in Riemann's Habilitationsvortrag that the general concept of a manifold first appeared [Rie13].[3] He proposed that geometry should study "extended magnitude or quantity," objects made of points with a continuous transition from one to another. To be mathematically useful, these objects should have sufficiently many functions so that it is possible to find coordinate functions which specify points uniquely, at least locally. One example he had in mind is quite advanced even from our modern point of view: the *moduli space of Riemann surfaces of genus g with n marked points.*

A Riemann surface is a compact one-dimensional complex manifold, as above. It is a rather deep result that all of these are algebraic, that is, cut out by polynomial equations in a complex projective space. Riemann's idea was that deformations of a Riemann surface structure on a fixed surface of genus g with n marked points as pictured in Figure 2.5 are uniquely specified (up to isomorphism) by $3g - 3 + n$ complex parameters. He wanted to use this to show that one can organize all such Riemannn surfaces into (something like) a $(6g - 6 + 2n)$-dimensional smooth manifold, each complex parameter giving rise to two dimensions [Loo00], so that you could study all Riemann surfaces *at the same time.* This has proven wildly successful, with entire fields doing dynamics and geometry on such moduli spaces.

We are far from having the theory to make this precise, but this example holds an important lesson: unlike spheres, many examples of smooth manifolds do *not* arise as subsets of some Euclidean space.

2.4 Problems

Problem 2 (Point-set topology of subspaces)**.**

(a) Prove that every subspace of a Hausdorff space is Hausdorff.

(b) Prove that every subspace of a second-countable space is second-countable.

Problem 3 (Connected vs. path components)**.** Prove that for a topological manifold, connected components coincide with path components.

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[3] You can read it at `https://www.emis.de/classics/Riemann/Geom.pdf`. More about the history of manifolds can be found in [Sch99].

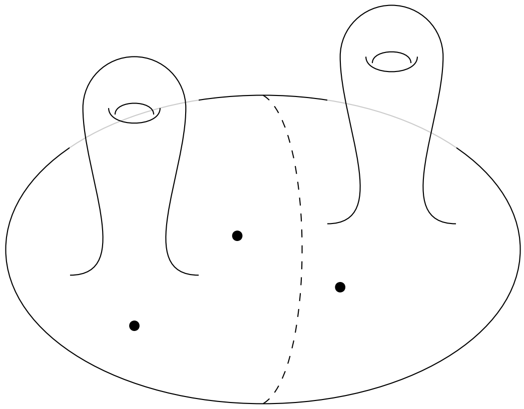


Figure 2.5 A surface of genus $g = 2$ with $n = 3$ marked points.

Problem 4 (Gluing smooth structures)**.** Suppose that if $U, V \subset X$ is an open cover of a second-countable Hausdorff space, and that we are given smooth atlases on U and V which agree on $U \cap V$. Prove that there exists a unique smooth maximal atlas on X which is compatible with the given ones on U and V.

Problem 5 (Complex projective plane)**.** There is a complex analogue of the real projective plane $\mathbb{R}P^k$, as constructed in the homework. The *complex projective plane* $\mathbb{C}P^k$ has points given by complex lines in $\mathbb{C}^{k+1}$, or equivalently by the quotient

$$(\mathbb{C}^{k+1}\backslash\{0\})/\sim$$

where $\sim$ is the equivalence relation generated by $(z_0, \ldots, z_k) \sim (\lambda z_0, \ldots, \lambda z_n)$ for $\lambda \in \mathbb{C}\backslash\{0\}$. Give $\mathbb{C}P^k$ a $2k$-dimensional smooth atlas.

Problem 6. Recall that the *quaternions* $\mathbb{H}$ are the 4-dimensional non-commutative unital $\mathbb{R}$-algebra with generators i, j, k and relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji, \ ik = -ki, \ jk = -kj$$

$$ij = k, \ jk = i, \ ki = j.$$

In analogy with the previous exercise, construct the *quaternionic projective plane* $\mathbb{H}P^k$.[4] What is its dimension?

[4] There is even an *octionic projective space* $\mathbb{O}P^2$, also known as the *Cayley projective plane,* but no $\mathbb{O}P^k$ for $k > 2$. This is harder to construct.

Chapter 3

Submanifolds

In the previous lecture we defined smooth manifolds, and we now discuss smooth *submanifolds*. We will use some results from multivariable calculus to produce examples of submanifolds of Euclidean spaces. I will assume you know the relevant results, but if you do not, you can find these in Chapters 3 & 4 of [DK04a]. After that we will give five constructions of the 2-torus.

3.1 Submanifolds

A loop of string in $\mathbb{R}^3$ can be thought of as a subset S of $\mathbb{R}^3$. Which subsets S describe such loops of string? Let us abstract the situation by declaring that the string is infinitely thin and bendable, but can not make sharp corners. Certainly an ordinary circle $\{(x, y, z) \mid x^2 + y^2 + z^2 = r^2\} \subset \mathbb{R}^3$ describes a loop of string, but so do many other subsets. Some differ from the circle by being more wiggly, and some by being knotted, see Figure 3.1.

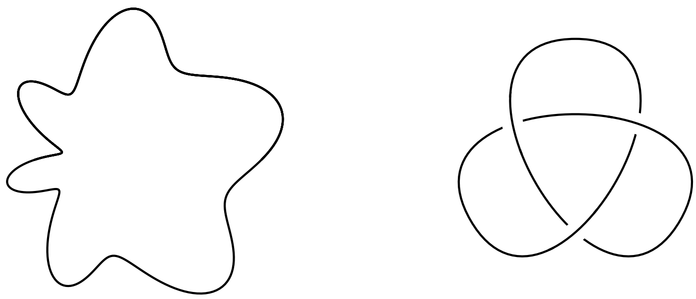


Figure 3.1 Some subsets of $\mathbb{R}^3$ which describe strings.

However, in spite of their complicated global behaviour, all locally look like smooth line segments: they are one-dimensional smooth submanifolds of $\mathbb{R}^3$, subsets of $\mathbb{R}^3$ that locally looks like $\mathbb{R}$. This illustrates why the study of smooth manifolds is so interesting: they have a straightforward local structure, but a rich global structure.

Of course, we need not restrict ourselves to one-dimensional objects: 2-spheres, 2-tori, and the surface of a coffee mug all locally look like $\mathbb{R}^2$. Indeed, for any $r \geqslant 0$ we will define r-dimensional smooth submanifolds as subsets of $\mathbb{R}^k$ that locally look like $\mathbb{R}^r$. More generally, we use charts to replace the ambient space $\mathbb{R}^k$ by a k-dimensional smooth manifold N.

3.1.1 The definition

To make precise the definition of a submanifold of a manifold, we recall the definitions from the previous lecture. A k-dimensional topological manifold is a second-countable Hausdorff space X which is locally homeomorphic to an open subset of $\mathbb{R}^k$. To give this the structure of k-dimensional smooth manifold, we need to provide the additional data of a maximal k-dimensional smooth atlas. This is a collection $(U_\alpha, V_\alpha, \phi_\alpha)$ of homeomorphisms $\phi_\alpha \colon \mathbb{R}^k \supset U_\alpha \to V_\alpha \subset X$ such that (i) $\bigcup_\alpha V_\alpha = X$, and (ii) all transition functions $\phi_\beta^{-1} \circ \phi_\alpha$ are smooth maps between open subsets of $\mathbb{R}^k$.

Intuitively, a submanifold is a manifold which lives inside another manifold. This is made precise by demanding it looks like a linear subspace of Euclidean space with respect to the atlas.

Definition 3.1.1. Let N be a k-dimensional smooth manifold. A subset $X \subset N$ is an *r-dimensional submanifold* if for each $p \in X$ there is a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ of N around p such that $\phi_\alpha^{-1}(X) = U_\alpha \cap \mathbb{R}^r$.

If X is a submanifold, it comes with a canonical structure of an r-dimensional smooth manifold. Firstly, X with the subspace topology is second countable and Hausdorff. We produce an atlas on this by taking a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ for N as above, and creating from it a chart $(U'_\alpha, V'_\alpha, \phi'_\alpha)$ for X as follows:

$$U'_\alpha := U_\alpha \cap \mathbb{R}^r, \quad V'_\alpha := X \cap V_\alpha, \quad \text{and} \quad \phi'_\alpha := \phi_\alpha|_{U'_\alpha}.$$

3.2 Examples of submanifolds using calculus

We for now concentrate on submanifolds of Euclidean space, and apply tools from multivariable calculus. We will eventually generalise these tools to manifolds, the philosophy being that differential topology is globalised multivariable analysis.

3.2.1 S^n by equations

Last chapter we defined the n-sphere by equations

$$S^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^n x_i^2 = 1 \right\},$$

and by hand gave a smooth atlas for it.

However, when you define a manifold by equations, it is much easier to obtain the smooth atlas using results from multivariable calculus; the *inverse function theorem*. This uses the notion of a total derivative of a map $g \colon \mathbb{R}^n \to \mathbb{R}^p$

(or between open subsets thereof) [DK04a, Section 4.5]: at $x \in \mathbb{R}^n$, the total derivative Dg_x of g at x is the linear map described by the $(p \times n)$-matrix of partial derivatives

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x) & \frac{\partial g_1}{\partial x_2}(x) & \cdots & \frac{\partial g_1}{\partial x_n}(x) \\ \frac{\partial g_2}{\partial x_1}(x) & \frac{\partial g_2}{\partial x_2}(x) & \cdots & \frac{\partial g_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1}(x) & \frac{\partial g_p}{\partial x_2}(x) & \cdots & \frac{\partial g_p}{\partial x_n}(x) \end{bmatrix}.$$

The local version of the inverse function theorem then says the following [DK04a, Theorem 3.2.4]:

Theorem 3.2.1 (Inverse function theorem)**.** *Let $U_0 \subset \mathbb{R}^n$ be open and $a \in U_0$. Suppose $g\colon U_0 \to \mathbb{R}^n$ is a smooth map whose total derivative Dg_a at a is an invertible linear map. Then there exists an open neighborhood $U \subset U_0$ of a such that $g(U)$ is open and*

$$g|_U : U \longrightarrow g(U)$$

is a diffeomorphism onto this open subset.

By adding variables, you can deduce the *implicit function theorem* [DK04a, Theorem 3.5.1] from this. The following is a consequence of that result [DK04a, Section 4.5]:

Theorem 3.2.2 (Submersion theorem)**.** *Let $U_0 \subset \mathbb{R}^n$ be open and $a \in U_0$. Suppose $g\colon U_0 \to \mathbb{R}^p$, $p \leqslant n$ is a smooth map whose total derivative Dg_a of g at a is a surjective linear map. Then there exist open neighborhoods $U \subset U_0$ of a and $V \subset \mathbb{R}^p$ of $g(a)$, and diffeomorphisms $\psi\colon \mathbb{R}^n \to U$ and $\varphi\colon \mathbb{R}^p \to V$, such that*

(i) $\psi(0) = a$,

(ii) $\varphi(0) = g(a)$, and

(iii) the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\cong", "\psi"'] \arrow[d, "\pi_p"'] & U \subset U_0 \subset \mathbb{R}^n \arrow[d, "g"] \\ \mathbb{R}^p \arrow[r, "\cong", "\varphi"'] & V \subset \mathbb{R}^p, \end{tikzcd}$$

with π_p the projection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_p)$. That is,

$$g(\psi(x_1, \ldots, x_n)) = \varphi(x_1, \ldots, x_p).$$

Remark 3.2.3. A stronger version of this theorem, which is the one stated as [DK04a, Theorem 4.5.2(iv)], says that φ can be taken to be translation near 0.

Parts (i) and (ii) are just normalisations, part (iii) is where the magic happens: the diffeomorphism ψ restricted to $\{0\} \times \mathbb{R}^{n-p} \subset \mathbb{R}^n$ gives a local parametrisation of the inverse image $g^{-1}(g(x))$ around x, identifying it with an open subset of the origin in $\mathbb{R}^{n-p}$. We conclude that the subset $g^{-1}(c)$ for $c \in \mathbb{R}^p$ is an $(n-p)$-dimensional smooth submanifold of $\mathbb{R}^n$ when each of the total derivatives Dg_x for $x \in g^{-1}(c)$ is surjective.

Example 3.2.4. If we take

$$g\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$
$$(x_0, \ldots, x_n) \longmapsto x_0^2 + \ldots + x_n^2,$$

and $c \neq 0 \in \mathbb{R}$, then the total derivative at $x = (x_0, \ldots, x_n)$ satisfying $x_0^2 + \ldots + x_n^2 = c$ is given by the $(1 \times n)$-matrix

$$\begin{bmatrix} 2x_0 & 2x_1 & \cdots & 2x_n \end{bmatrix}$$

with not all x_i zero. If $c \neq 0$, then not all entries can vanish at the same time and this matrix is surjective. In particular, we can take $c = 1$ to obtain another proof that the n-sphere is a smooth manifold.

Example 3.2.5. Let p, q be positive integers, and take

$$g\colon \mathbb{C}^2 \longrightarrow \mathbb{C} \times \mathbb{R}$$
$$(z_1, z_2) \longmapsto (z_1^p + z_2^q, |z_1|^2 + |z_2|^2).$$

This is smooth, and its total derivative is surjective at all points $g^{-1}(0, \epsilon)$ for $(0, \epsilon) \in \mathbb{C} \times \mathbb{R}$ with $\epsilon > 0$ small enough. Thus the inverse image $g^{-1}(0, \epsilon)$ is a one-dimensional submanifold of $\mathbb{C}^2$, which lies inside S^3_ϵ, the sphere of radius ϵ around the origin. It is in fact also a one-dimensional submanifold of S^3_ϵ, and if we remove a point from it and identify the result with $\mathbb{R}^3$, the result is a so-called (p, q)-*torus link*. See Figure 3.2 for an example.

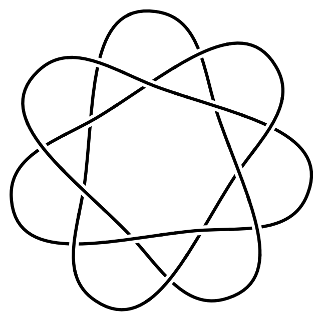


Figure 3.2 A $(3, 7)$-torus knot (since 3 and 7 are coprime, there is only a single component).

3.2.2 S^n by parametrisations

One can often parametrise solution sets of equations, e.g. S^1 is the image of

$$h\colon \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$\theta \longmapsto (\cos(\theta), \sin(\theta)).$$

This map is not a bijection, but it is locally a bijection. It seems quite plausible that it is in fact a local diffeomorphism of $\mathbb{R}$ onto S^1, though giving an explicit formula may be hard. However, the difficulty of finding explicit formula's can be avoided by using the inverse function theorem again, in a slightly different guise [DK04a, Section 4.3].

Theorem 3.2.6 (Immersion theorem). *Let $U_0 \subset \mathbb{R}^p$ be an open subset and $a \in U_0$. Suppose $h\colon U_0 \to \mathbb{R}^n$, $p \leqslant n$, is a smooth map whose total derivative Dh_a of h at a is injective. Then there exist open neighborhoods $U \subset U_0$ of a and $V \subset \mathbb{R}^n$ of $h(a)$, and diffeomorphisms $\psi\colon \mathbb{R}^p \to U$ and $\varphi\colon \mathbb{R}^n \to V$, such that*

(i) $\psi(0) = a$,

(ii) $\varphi(0) = h(a)$, and

(iii) the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "\cong", "\psi"'] \arrow[d, "\iota_p"'] & U \subset \mathbb{R}^p \arrow[d, "h"] \\ \mathbb{R}^n \arrow[r, "\cong", "\varphi"'] & V \subset \mathbb{R}^n, \end{tikzcd}$$

with ι_p the inclusion $(x_1, \ldots, x_p) \mapsto (x_1, \ldots, x_p, 0, \ldots, 0)$. That is,

$$h(\psi(x_1, \ldots, x_p)) = \varphi(x_1, \ldots, x_p, 0, \ldots, 0).$$

Remark 3.2.7. As before, there is a stronger version stated as [DK04a, Theorem 4.3.1] which says that ψ can be taken to be translation near 0.

Again part (iii) is the important part: it provides a chart for $h(U)$ as in the definition of a submanifold. We will later see that the image of h is a submanifold if we not only suppose that its derivative is injective everywhere but also that the map h is a homeomorphism onto its image.

Example 3.2.8. If we want to parametrise the n-sphere S^n, we will need more than one function h_i. For example, we can use $2(n+1)$ ones indexed by $0 \leqslant i \leqslant n$ and a sign ± 1:

$$\begin{aligned} h_i^{\pm}\colon \{y \in \mathbb{R}^n \mid ||y|| < 1\} &\longrightarrow \mathbb{R}^{n+1} \\ (y_1, \ldots, y_n) &\longmapsto (y_1, \ldots, y_{i-1}, \pm\sqrt{1 - ||y||^2}, y_i, \ldots, y_n). \end{aligned}$$

Each covers one of the two hemispheres in each of the $n+1$ directions of $\mathbb{R}^{n+1}$.

3.3 Five constructions of the 2-torus

Another important example of a smooth manifold is the 2-torus, one of the basic surfaces. We will now give five constructions of the torus,

(1) By specifying it as a submanifold of $\mathbb{R}^3$ using equations.

(2) By parametrising it as a submanifold of $\mathbb{R}^3$.

(3) As a product of two circles.

(4) By gluing edges of a square $[0, 1]^2$.

(5) As a quotient $\mathbb{R}^2/\mathbb{Z}^2$.

All these constructions give us diffeomorphic smooth manifolds, but we will not prove this. The first three can be thought of as naturally being subsets of some Euclidean spaces, but the underlying topological space of a smooth manifold obtained by gluing or quotients is not naturally a subset of a Euclidean space. This is one of the reasons we gave an abstract definition of manifold in the last chapter.

3.3.1 The 2-torus specified by equations

Our first construction of a 2-torus is as those points that are distance 1 from a circle of radius $\sqrt{2}$: it consists of those points (x, y, z) in $\mathbb{R}^3$ satisfying the equation $(2 - \sqrt{x^2 + y^2})^2 + z^2 = 1$. More precisely, define

$$\begin{aligned} g\colon \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto \left(2 - \sqrt{x^2 + y^2}\right)^2 + z^2. \end{aligned}$$

This is smooth and has surjective total derivative at all points in the pre-image of 1. Thus the submersion theorem tells us that $g^{-1}(1)$ is a two-dimensional smooth submanifold of $\mathbb{R}^3$:

$$\mathbb{T}^2 = g^{-1}(1).$$

3.3.2 The 2-torus parametrised

We can parametrise the 2-torus, defined as $g^{-1}(1) \subset \mathbb{R}^3$, as the image of

$$\begin{aligned} h\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (\theta, \phi) &\longmapsto [(2 + \cos(\theta))\cos(\phi), (2 + \cos(\theta))\sin(\phi), \sin(\theta)]. \end{aligned}$$

This is smooth and has injective total derivative at all points in its domain. Thus the immersion theorem provides local charts for the image of h. These exhibit the image of h as a submanifold of $\mathbb{R}^3$, and give another description of the 2-torus as a two-dimensional smooth submanifold of $\mathbb{R}^3$:

$$\mathbb{T}^2 = \operatorname{im}(h).$$

Some care is required now, as h is not a homeomorphism onto its image because it is not injective. Trying to amend this leads one to the definition of the 2-torus by gluing or as a quotient.

3.3.3 The 2-torus as a product

There is a general method to produce new submanifolds out of old ones.

Lemma 3.3.1. *Suppose that $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are submanifolds of dimensions p and q respectively. Then $X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ is a $(p+q)$-dimensional submanifold of $\mathbb{R}^{n+m}$.*

Sketch of proof. Local parametrisations of X near x and Y near y combine a local parametrisation of $X \times Y$ near (x, y). $\square$

This gives a different construction of $\mathbb{T}^2$ as a submanifold of $\mathbb{R}^4$: take the product of $S^1 \subset \mathbb{R}^2$ with itself. Of course, we can forget that S^1 is a submanifold of $\mathbb{R}^2$, and instead take the abstract product of manifolds discussed in the previous lecture:

$$\mathbb{T}^2 = S^1 \times S^1.$$

3.3.4 The 2-torus by gluing

Let us take a square $[0, 1]^2$ and make identifications along its boundary $\partial[0, 1]^2 = \{(x, y) \in [0, 1]^2 \mid x \in \{0, 1\} \text{ or } y \in \{0, 1\}\}$ as in Figure 3.3: take $[0, 1]^2/\sim$ with $\sim$ the equivalence relation generated by

$$(0, y) \sim (1, y) \qquad \text{and} \qquad (x, 0) \sim (x, 1).$$

That is, the left edge $\{0\} \times [0, 1]$ gets identified with right edge $\{1\} \times [0, 1]$ and the bottom edge $[0, 1] \times \{0\}$ with the top edge $[0, 1] \times \{1\}$ Such a gluing of the square produces a torus.

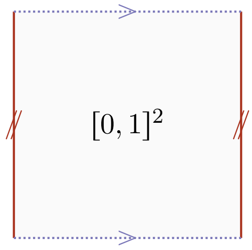


Figure 3.3 The 2-torus is obtained by identifying edges of $[0, 1]^2$.

We now give a 2-dimensional smooth atlas on $[0, 1]^2/\sim$, see Figure 3.4. It is easy to give charts for a point represented by $(x, y) \in (0, 1)^2$; just use a small open disk $B_\epsilon(x, y)$ contained in $(0, 1)^2$. For equivalence classes $[(x, 0)]$ represented by $(x, 0)$ with $x \in (0, 1)$ we use the chart determined by

$$\begin{aligned} \phi\colon B_\epsilon(x, 0) &\longrightarrow [0, 1]^2/\sim \\ (x', y') &\longmapsto \begin{cases} [(x', y' + 1)] & \text{if } y' < 0, \\ [(x', y')] & \text{if } y' \geqslant 0, \end{cases} \end{aligned}$$

and similarly for the element represented by $(0, y)$ with $y \in (0, 1)$. For the equivalence class $[(0, 0)]$ we use the chart determined by

$$\begin{aligned} \phi\colon B_\epsilon(0, 0) &\longrightarrow [0, 1]^2/\sim \\ (x', y') &\longmapsto \begin{cases} [(x' + 1, y' + 1)] & \text{if } y' < 0,\ x' < 0, \\ [(x' + 1, y')] & \text{if } x' < 0, \\ [(x', y' + 1)] & \text{if } y' < 0, \\ [(x', y')] & \text{otherwise.} \end{cases} \end{aligned}$$

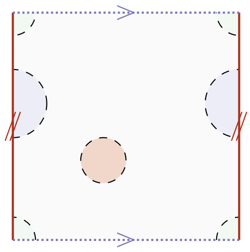


Figure 3.4 The open subsets V_α for three charts, one of each type.

The transition functions are mostly given by the identity map which is obviously smooth, but sometimes by a translation which is also obviously smooth. See Figure 3.5 for the hardest case. We conclude that

$$\mathbb{T}^2 = [0,1]^2/\sim.$$

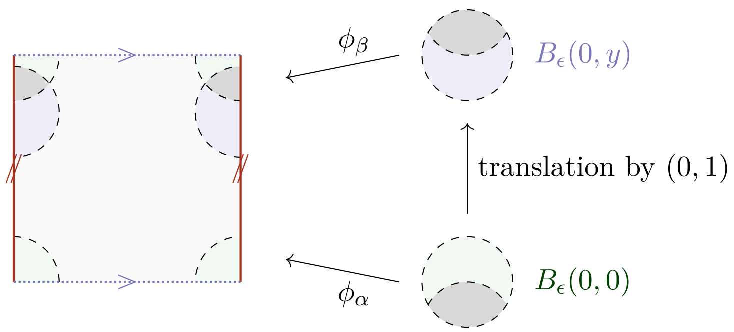


Figure 3.5 A transition function.

The lesson is, using terms we have not defined yet: *a sufficiently nice gluing of a k-dimensional manifold with corners along its boundary is again a k-dimensional manifold.* In the above example $k = 2$, the manifold with corners is $[0,1]^2$ and the boundary is $\partial[0,1]^2$.

Example 3.3.2. Changing the identifications to those in Figure 3.6 and using similar charts we can endow the Klein bottle and real projective plane with a 2-dimensional smooth structure.

3.3.5 The 2-torus as a quotient

Let us recast this definition in terms of group theory. If you are not familiar with group theory, you should take a look at a textbook on it, e.g. [Arm88].

We can add elements of $\mathbb{R}^2$ from which we obtain an action of the abelian group $\mathbb{Z}^2$ on $\mathbb{R}^2$: the element $(n, m) \in \mathbb{Z}^2$ acts on (x, y) by sending it to its translate $(n, m) \cdot (x, y) := (x + n, y + m)$. Let us look at the set

$$\mathbb{R}^2/\mathbb{Z}^2 := \mathbb{R}^2/\sim \text{ with } (x, y) \sim (x', y') \text{ if } (n, m) \cdot (x, y) = (x', y') \text{ for } (m, n) \in \mathbb{Z}^2$$

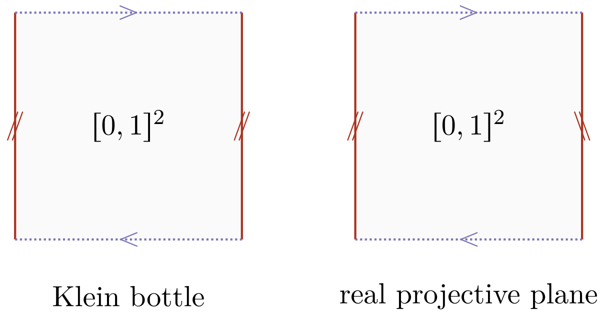


Figure 3.6 Two more 2-dimensional smooth manifolds obtained by identifying edges of $[0,1]^2$.

with the quotient topology. This is still Hausdorff and second countable.

We claim that $\mathbb{R}^2/\mathbb{Z}^2$ inherits from $\mathbb{R}^2$ the structure of a 2-dimensional smooth manifold. To do so we describe a 2-dimensional smooth atlas on $\mathbb{R}^2/\mathbb{Z}^2$: for a point $(x,y) \in \mathbb{R}^2$ we can consider the open disks $B_\epsilon(x,y)$ for $\epsilon < \frac{1}{4}$. The composition of the inclusion with the quotient map

$$B_\epsilon(x,y) \hookrightarrow \mathbb{R}^2 \xrightarrow{q} \mathbb{R}^2/\mathbb{Z}^2$$

is injective as $\epsilon < \frac{1}{4}$. We denote its image by $V^\epsilon_{(x,y)}$ and resulting map by

$$\phi^\epsilon_{(x,y)} \colon B_\epsilon(x,y) \longrightarrow V^\epsilon_{(x,y)}.$$

We claim these chart give an atlas. Since the map $q\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is surjective, the $V^\epsilon_{(x,y)}$ cover. For any two open subsets $V^\epsilon_{(x,y)}$, $V^{\epsilon'}_{(x',y')}$, the transition function is just given by translation and hence is smooth.

One way to visualise the result is to give a *fundamental domain*: an open subset $U \subset \mathbb{R}^n$ such that $U \to \mathbb{R}^2/\mathbb{Z}^2$ is injective and $\bar{U} \to \mathbb{R}^2/\mathbb{Z}^2$ is surjective. Then you can think of $\mathbb{R}^2/\mathbb{Z}^2$ as being obtained from $\bar{U}$ by making identifications along ∂U. In this case a moment's reflection produces $(0,1)^2 \subset \mathbb{R}^2$ as a candidate; no two elements differ by translation by $(m,n) \in \mathbb{Z}^2$ so $(0,1)^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is injective, but $(x,y) \sim (x - \lfloor x \rfloor, y - \lfloor y \rfloor) \in [0,1]^2$ so $[0,1]^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is surjective. Thus $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to $[0,1]^2/\sim$ as in the previous section, and thus we have produced another description of the 2-torus. Under this identification, the charts we have described to go the charts in the previous section. We get

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2.$$

There is a general lesson here: *a quotient of a k-dimensional smooth manifold by a sufficiently nice action of a discrete group G is again a k-dimensional smooth manifold.* In the above example $k = 2$, the manifold is $\mathbb{R}^2$ and $G = \mathbb{Z}^2$.

Example 3.3.3. Can we come up with other examples? One idea would be to use with some subgroup G of $\mathbb{Z}^2$, and take

$$\mathbb{R}^2/G := \mathbb{R}^2/\sim \text{ with } (x,y) \sim (x',y') \text{ if } g \cdot (x,y) = (x',y') \text{ for } g \in G,$$

instead of $\mathbb{R}^2/\mathbb{Z}^2$. Most of these seem to give variations on the 2-torus, but the subgroup $\mathbb{Z} \times \{0\} \subset \mathbb{Z}^2$ does *not*. In this case a fundamental domain is given by $(0, 1) \times \mathbb{R}$, and $\mathbb{R}^2/(\mathbb{Z} \times \{0\})$ is given by identifying the left edge $\{0\} \times \mathbb{R}$ of the infinite strip $[0, 1] \times \mathbb{R}$ with the right edge $\{1\} \times \mathbb{R}$; an infinite *cylinder*.

Chapter 4

Smooth maps

In this lecture we will define smooth maps. This material appears at the end of Section 1 of [BJ82], as well as Section 2. For more details, see [Tu11, Chapters 6, 8].

4.1 Smooth maps and diffeomorphisms

Let us recall some definitions from Lecture 2, on which we shall elaborate now:

Definition 4.1.1. Let M and N be smooth manifolds of dimension m and n, with smooth maximal atlases $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ and $\{(U'_\beta, V'_\beta, \phi'_\beta)\}$. A continuous map $f\colon M \to N$ is said to be *smooth* if for all charts $(U_\alpha, V_\alpha, \phi_\alpha)$ of M and $(U'_\beta, V'_\beta, \phi'_\beta)$ of N, the map

$$(\phi'_\beta)^{-1} \circ f \circ \phi_\alpha\colon \mathbb{R}^m \supset \phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_\beta)) \longrightarrow (\phi'_\beta)^{-1}(V'_\beta) = U'_{\beta'} \subset \mathbb{R}^n \tag{4.1}$$

between open subsets of Euclidean spaces is smooth.

It may be helpful to expand (4.1) into a commutative diagram

$$\begin{tikzcd} \mathbb{R}^m \supset \phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_\beta)) \arrow[r, "\phi_\alpha", "\cong"'] \arrow[d, "(\phi'_\beta)^{-1}\circ f\circ\phi_\alpha"'] & V_\alpha \cap f^{-1}(V'_\beta) \subset M \arrow[d, "f"] \\ \mathbb{R}^n \supset U'_{\beta'} = (\phi'_\beta)^{-1}(V'_\beta) \arrow[r, "\phi'_\beta", "\cong"'] & V'_\beta \subset N. \end{tikzcd}$$

In terms the definition of a smooth map, we explained when we consider two smooth manifolds to be the same:

Definition 4.1.2. A smooth map $g\colon M \to N$ between smooth manifolds is a *diffeomorphism* if it has a smooth inverse.

We say M and N are *diffeomorphic* if there is a diffeomorphism between them. This is an equivalence relation.

Example 4.1.3. The real projective space $\mathbb{R}P^1$ is diffeomorphic to S^1.

Example 4.1.4. The complex projective plane $\mathbb{C}P^1$ is diffeomorphic to S^2.

Example 4.1.5. All five definitions of $\mathbb{T}^2$ that we gave—by equations, by parametrization, as a product, by gluing, and a quotient—are diffeomorphic.

Example 4.1.6. $\mathbb{R}^k$ is diffeomorphic to $\mathbb{R}^l$ if and only if $k = l$. This is a smooth variant of *invariance of domain*.

4.1.1 Properties of smooth maps

Definition 4.1.1 at first sight involves a condition that is hard to check, as both maximal atlases will in general have infinitely many charts. However, it suffices to only verify the condition on a smaller collection of charts; all these need to do is cover the entire domain M, as well as the image $f(M) \subset N$ in the target.

Lemma 4.1.7. *Let $\{(U_i, V_i, \phi_i)\}_{i\in I}$ and $\{(U'_j, V'_j, \phi'_j)\}_{j\in J}$ be collections of charts of M and N respectively, such that $\bigcup_{i\in I} V_i = M$ and $f(M) \subset \bigcup_{j\in J} V'_j$. If for all $i \in I$ and $j \in J$, the map*

$$(\phi'_j)^{-1} \circ f \circ \phi_i \colon \mathbb{R}^m \supset \phi_i^{-1}(V_i \cap f^{-1}(V'_j)) \longrightarrow (\phi'_j)^{-1}(V'_j) = U'_j \subset \mathbb{R}^n$$

between open subsets of a Euclidean space is smooth, then f is smooth.

Proof. We must prove that every map

$$(\phi'_\beta)^{-1} \circ f \circ \phi_\alpha \colon \mathbb{R}^m \supset \phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_\beta)) \longrightarrow (\phi'_\beta)^{-1}(V'_\beta) = U'_{\beta'} \subset \mathbb{R}^n$$

is smooth. Since being smooth is a local property, it is enough to prove that each $x \in \phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_\beta))$ has an open neighbourhood such that the restriction to this open neighbourhood is smooth. Let us pick charts (U_i, V_i, ϕ_i) so that $x \in V_i$ and (U'_j, V'_j, ψ'_j) so that $f(x) \in V'_j$. Then we can write the restriction to $\phi_\alpha^{-1}(V_\alpha \cap V_i \cap (f^{-1}(V'_\beta \cap V'_j))$ as

$$((\phi'_\beta)^{-1} \circ \phi'_j) \circ ((\phi'_j)^{-1} \circ f \circ \phi_j) \circ (\phi_i^{-1} \circ \phi_\alpha),$$

which is a composition of three smooth functions. □

A first consequence of this is that in Definition 4.1.1 we could have equivalently taken *any* atlases of M and N compatible with their maximal atlases. A second consequence is the following rephrasing:

Corollary 4.1.8. *A map $f \colon M \to N$ is smooth if and only if for all $m \in M$ there is a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ around m in M and a chart $(U'_\beta, V'_\beta, \phi'_\beta)$ around $f(m)$ in N, such that the map*

$$(\phi'_\beta)^{-1} \circ f \circ \phi_\alpha \colon \mathbb{R}^m \supset \phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_\beta)) \longrightarrow (\phi'_\beta)^{-1}(V'_\beta) = U'_\beta \subset \mathbb{R}^n$$

between open subsets of Euclidean spaces, is smooth at m.

Sometimes you can pick a few charts particularly well-suited to your situation:

Example 4.1.9. A map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is smooth in the above sense if and only if it is smooth in the sense of multivariable calculus, since we may use the identity as a single chart for both $\mathbb{R}^m$ and $\mathbb{R}^n$. This justifies our lack of distinction between "smooth in the sense of multivariable calculus" and "smooth in the sense of differential topology."

Example 4.1.10. If M and N are spheres S^m and S^n, we know that each of them can be covered by two charts using stereographic projection and hence we can get away with checking only four cases.

Example 4.1.11. The diagonal map

$$\begin{aligned} \Delta\colon M &\longrightarrow M \times M \\ p &\longmapsto (p,p) \end{aligned}$$

is smooth, where the target is made into a smooth manifold by taking products of charts. Indeed, we can verify this using charts $(U_\alpha, V_\alpha, \phi_\alpha)$ on the domain and charts of the form $(U_\alpha \times U_\alpha, V_\alpha \times V_\alpha, \phi_\alpha \times \phi_\alpha)$ on the target. The result then amounts to verifying that the diagonal $\mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k$ given by $x \mapsto (x,x)$ is smooth.

4.2 Constructing smooth maps

In practice, one often constructs new smooth maps out of old ones using one of the following tools. Parts (iii) and (iv) use the construction of a smooth structure on an open subset of a smooth manifold.

Lemma 4.2.1.

(i) For every smooth manifold M, the identity map id_M is smooth.

(ii) If $\{U_i\}$ is an open cover of M and each $f|_{U_i}\colon U_i \to N$ is smooth, then $f\colon M \to N$ is smooth.

(iii) If $f\colon M \to N$ and $g\colon N \to P$ are smooth, then so is $g \circ f\colon M \to P$.

(iv) If $f\colon M \to N$ is smooth and $U \subset M$ is open, then $f|_U\colon U \to N$ is smooth.

Note that (iv) gives the converse to (ii), so we can replace "if" by "if and only if" there.

Proof. (i) If $f = \mathrm{id}_M$, then (4.1) becomes

$$\phi_\beta^{-1} \circ \phi_\alpha\colon \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \longrightarrow \phi_\beta^{-1}(V_\beta),$$

which is smooth by definition of an atlas, as it is a transition function followed by the inclusion of an open subset.

(ii) By Lemma 4.1.7, it is enough to verify smoothness with respect to the collection of charts $(U_\alpha, V_\alpha, \phi_\alpha)$ with the property that $U_\alpha \subset U_i$ for some i. In that case, we can replace in (4.1) the map f by $f|_{U_i}$ and smoothness follows from the hypothesis that $f|_{U_i}$ is smooth.

(iii) We write out (4.1) as

$$(\phi''_\gamma)^{-1}\circ g\circ f\circ\phi_\alpha\colon \mathbb{R}^m \supset \phi_\alpha^{-1}(V_\alpha\cap(g\circ f)^{-1}(V''_\gamma)) \longrightarrow (\phi''_\gamma)^{-1}(V''_\gamma) = U''_\gamma \subset \mathbb{R}^p$$

Then for each chart $(U'_\beta, V'_\beta, \phi'_\beta)$ we can write $(\phi''_\gamma)^{-1}\circ g\circ f\circ\phi_\alpha$ as

$$((\phi''_\gamma)^{-1}\circ g\circ\phi'_\beta)\circ((\phi'_\beta)^{-1}\circ f\circ\phi_\alpha)$$

when restricting to $\phi_\alpha^{-1}(V_\alpha\cap f^{-1}(V'_\beta)\cap(g\circ f)^{-1}(V''_\gamma))$. This is a composition of a smooth map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ with a smooth map between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, and hence is smooth. Since the open subsets $\phi_\alpha^{-1}(V_\alpha\cap f^{-1}(V'_\beta)\cap(g\circ f)^{-1}(V''_\gamma))$ give an open cover of $\phi_\alpha^{-1}(V_\alpha\cap(g\circ f)^{-1}(V''_\gamma))$ and smoothness is a local property, this tells us that $((\phi''_\gamma)^{-1}\circ g\circ f\circ\phi_\alpha)$ is smooth.

(iv) It suffices to prove that the inclusion $i_U\colon U\to M$ is smooth, as then $f|_U$ is the composition $f\circ i_U$ of two smooth maps. Using the chart on U obtained by restricting those on M, (4.1) becomes

$$\phi_\beta^{-1}\circ\phi_\alpha\colon \phi_\alpha^{-1}(U\cap V_\alpha\cap V_\beta)\longrightarrow\phi_\beta^{-1}(V_\alpha\cap V_\beta),$$

which is just the restriction of the smooth map $\phi_\beta^{-1}\circ\phi_\alpha$ to an open subset. □

Remark 4.2.2. Using part (i) and (iii) we can define a *1-category* Mfd of smooth manifolds; its objects are smooth manifolds and morphisms from M to N are smooth maps. Part (i) then implies that this category has identity morphisms and part (iii) implies that composition is well-defined. We takes this up again later.

Category theory is a useful language for studying topology and related fields, as many objects of interest can be defined in terms of *universal properties* saying how other objects should map to them or receive maps from them. Let us give two examples.

Recall that we have defined the disjoint union of $M\sqcup N$ of two manifolds of the same dimension. It is a consequence of parts (ii) and (iv) that a map $f\colon M\sqcup N\to P$ is smooth if and only if $f|_M$ and $f|_N$ are. This exhibits $M\sqcup N$ as the (categorical) coproduct in Mfd.

We also defined the product $M\times N$ of two smooth manifolds. By Problem 7 the projection $\pi_1\colon M\times N\to M$ and $\pi_2\colon M\times N\to N$ are smooth. Thus by (iii) if $f\colon P\to M\times N$ is smooth so are its components $\pi_1\circ f$ and $\pi_2\circ f$. Note that we can recover f as

$$P\xrightarrow{\Delta}P\times P\xrightarrow{(\pi_1\circ f)\times(\pi_2\circ f)}M\times N,$$

which is smooth as a consequence of (iii), Example 4.1.11, and Problem 7 (d). We conclude that $f\colon P\to M\times N$ is smooth if and only if its components $\pi_1\circ f\colon P\to M$ and $\pi_2\circ f\colon P\to N$ are. Thus $M\times N$ is the (categorical) product in Mfd.

It is particularly easy to construct smooth maps into or out of submanifolds.

Lemma 4.2.3. *Suppose that $X \subset M$ is a submanifold.*

(i) The inclusion $i\colon X \to M$ is a smooth map.

(ii) If $f\colon X \to N$ extends to a smooth map $\tilde{f}\colon M \to N$, then f is smooth.

(iii) If $g\colon N \to X$ is such that $i \circ g$ is smooth, then g is smooth.

Proof. (i) Since X is a submanifold, we can find charts $(U_\alpha, V_\alpha, \phi_\alpha)$ of M covering X such that $\phi_\alpha^{-1}(V_\alpha \cap X) = U_\alpha \cap \mathbb{R}^k$. In fact, it is these charts that generate the atlas on X. By Lemma 4.2.1 (ii), it suffices to prove that $i|_{V_\alpha \cap X}\colon V_\alpha \cap X \to M$ is smooth. Since we can cover $V_\alpha \cap X$ by the single chart $(U_\alpha \cap \mathbb{R}^k, V_\alpha \cap X, \phi_\alpha|_{U_\alpha \cap \mathbb{R}^k})$ and its image by the chart $(U_\alpha, V_\alpha, \phi_\alpha)$, by Lemma 4.1.7 it suffices to prove that

$$\phi_\alpha^{-1} \circ i|_{V_\alpha \cap X} \circ \phi_\alpha|_{U_\alpha \cap \mathbb{R}^k}\colon U_\alpha \cap \mathbb{R}^k \longrightarrow U_\alpha$$

is smooth. But it is just the inclusion of those points with last $m - k$ coordinates equal to 0, which is clearly smooth!

(ii) Since $f = \tilde{f} \circ i$, this follows from (i) and Lemma 4.2.1 (iii).

(iii) We again use charts $(U_\alpha, V_\alpha, \phi_\alpha)$ of M covering X such that $\phi_\alpha^{-1}(V_\alpha \cap X) = U_\alpha \cap \mathbb{R}^k$. By Lemma 4.1.7, g is smooth if and only if

$$(\phi_\alpha|_{U_\alpha \cap \mathbb{R}^k})^{-1} \circ g \circ \phi'_\beta$$

is smooth for all charts $(U'_\beta, V'_\beta, \phi'_\beta)$ of N. However, we are guaranteed that all maps

$$(\phi_\alpha)^{-1} \circ g \circ \phi'_\beta$$

are smooth, which differ from the previous maps by composition with the standard inclusion $\mathbb{R}^k \to \mathbb{R}^m$ onto the first k coordinates, $m \geqslant k$. Composing these with the projection $\mathbb{R}^m \to \mathbb{R}^k$ onto the first k coordinates, we recover the previous maps as a composition of smooth maps and hence they are smooth. $\square$

Remark 4.2.4. We will later be able to prove that (ii) is actually an "if and only if".

Example 4.2.5 (Rotations as diffeomorphisms of S^n). By Lemma 4.2.3 (ii) and (iii), a map $S^n \to S^n$ is smooth if it extends to a smooth map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. We will use this to construct diffeomorphisms of S^n. Let us take a matrix $A \in O(n+1)$, the group of orthogonal $(n+1) \times (n+1)$-matrices. By definition an orthogonal matrix preserves the Euclidean norm $||x||$, and hence $x \mapsto Ax$ sends S^n to S^n. Furthermore, each entry of Ax is just a linear combination of the entries of x so is easily seen to be smooth. Thus $x \mapsto Ax$ gives an example of a smooth map $S^n \to S^n$. It has an evident smooth inverse given by $x \mapsto A^{-1}x$.

We have thus just produced a map $O(n+1) \to \mathrm{Diff}(S^n)$, the latter the group of diffeomorphisms of S^n. The latter can be endowed with a natural topology which makes this map continuous. If $n \leqslant 3$, it is a homotopy equivalence by work of Smale and Hatcher [Sma59, Hat83]. If $n \geqslant 4$, it is not a homotopy equivalence; the case $n = 4$ was only proven recently [Wat18].

Example 4.2.6 (General linear groups). The set $M_n(\mathbb{R})$ of $(n \times n)$-matrices with real entries can be identified with $\mathbb{R}^{n^2}$, and through this identification can be made into a smooth n^2-dimensional manifold. Matrix multiplication gives a map

$$\begin{aligned} \mu\colon M_n(\mathbb{R}) \times M_n(\mathbb{R}) &\longrightarrow M_n(\mathbb{R}) \\ (A, B) &\longmapsto AB \end{aligned}$$

which we claim is smooth. To check this, we use that there is a single chart covering $M_n(\mathbb{R})$, the standard identification, and similarly a single chart covering $M_n(\mathbb{R}) \times M_n(\mathbb{R})$, a product of two standard identifications. By Lemma 4.1.7 it suffices to prove that matrix multiplication is smooth with respect to these charts only; this is true because it is a polynomial in the entries of the matrices and hence smooth.

The open subset $\mathrm{GL}_n(\mathbb{R}) \subset M_n(\mathbb{R})$ of invertible matrices, which can be described as the complement of the closed subset determined by the equation $\det = 0$, is hence also a smooth n^2-dimensional manifold. Since a composition of invertible matrices is again invertible, Lemma 4.2.3 implies that matrix multiplication restricts to a smooth map

$$\mu\colon \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R}).$$

We can also take the inverse of an invertible matrix, giving a map

$$\begin{aligned} \iota\colon \mathrm{GL}_n(\mathbb{R}) &\longrightarrow \mathrm{GL}_n(\mathbb{R}) \\ A &\longmapsto A^{-1}, \end{aligned}$$

which is also smooth. Indeed, using again the standard identifications as charts, we can use Cramer's rule:

$$A^{-1} = \frac{1}{\det(A)} C^T$$

with C the cofactor matrix; its (i,j)th entry is given by $(-1)^{i+j} \det(\hat{A}_{ij})$ where $\hat{A}_{ij}$ is obtained from A by deleting the ith row and jth column. The details are not important, only that it is a smooth function of the entries of an invertible matrix.

An example of a group which compatibly is a smooth manifold deserves a name:

Definition 4.2.7. A *Lie group* is a smooth manifold G which is also a group, such that multiplication $\mu\colon G \times G \to G$ and inverse $\iota\colon G \to G$ are both smooth.

4.3 Problems

Problem 7 (Maps in or out of products)**.** Let X, Y be smooth manifolds.

(a) Prove that the projection maps $\pi_1\colon X \times Y \to X$ given by $\pi_1(x, y) = x$ and $\pi_2\colon X \times Y \to Y$ given by $\pi_2(x, y) = y$ are both smooth.

(b) Show that

$$\begin{aligned} T_{(x,y)}(X \times Y) &\longrightarrow T_x X \oplus T_y Y \\ v &\longmapsto (d_{(x,y)}\pi_1(v), d_{(x,y)}\pi_2(v)) \end{aligned} \tag{4.2}$$

is an isomorphism of $\mathbb{R}$-vector spaces.

(c) Fixing a point $y \in Y$, there is an injection map

$$\begin{aligned} i_y \colon X &\longrightarrow X \times Y \\ x &\longmapsto (x, y), \end{aligned}$$

which you may assume is smooth. Prove that its derivative $d_x i_y \colon T_x X \to T_{(x,y)}(X \times Y) \cong T_x X \oplus T_y Y$ is given by $w \mapsto (w, 0)$.

(d) Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be smooth maps. Prove that

$$\begin{aligned} f \times g \colon X \times Y &\longrightarrow X' \times Y' \\ (x, y) &\longmapsto (f(x), g(y)) \end{aligned}$$

is smooth. Prove that its derivative $d_{(x,y)}(f \times g) \colon T_{(x,y)}(X \times Y) \to T_{(f(x),g(y))}(X' \times Y')$ is given by $(v, w) \mapsto (d_x f(v), d_y g(w))$ under the isomorphism (4.2).

Problem 8 (Complex general linear groups)**.** Show that $\mathrm{GL}_n(\mathbb{C})$ is a $(2n)^2$-dimensional Lie group.

Problem 9 (Orthogonal groups)**.** Show that $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$ is an $\frac{n(n-1)}{2}$-dimensional Lie group.

Chapter 5

Derivatives

In this lecture we will define the derivatives of a smooth map at a point, and in the next lecture we will assemble these together. This material appears at the end of Section 1 of [BJ82], as well as Section 2. For more details, see [Tu11, Chapters 6, 8].

5.1 Derivatives and tangent spaces

We want to extend the notion of a derivative of a smooth map between two open subsets of Euclidean space to a smooth map between smooth manifolds. This is useful because the derivative determines the local behaviour of smooth maps. Using it, we will be able to formulate and prove global versions of the submersion and immersion theorem.

If you are unfamiliar with the total derivative of smooth maps between open subsets of Euclidean spaces, take a look at Chapter 2 of [DK04a]. For each $x \in \mathbb{R}^k$, we can think of $\mathbb{R}^k$ as a space of vectors based at x. It has a standard basis. A smooth map $g\colon \mathbb{R}^k \supset U \to \mathbb{R}^{k'}$ has a total derivative at x given by the linear map, whose matrix with respect to the standard bases is the $(k' \times k)$-matrix of partial derivatives

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x) & \frac{\partial g_1}{\partial x_2}(x) & \cdots & \frac{\partial g_1}{\partial x_k}(x) \\ \frac{\partial g_2}{\partial x_1}(x) & \frac{\partial g_2}{\partial x_2}(x) & \cdots & \frac{\partial g_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{k'}}{\partial x_1}(x) & \frac{\partial g_{k'}}{\partial x_2}(x) & \cdots & \frac{\partial g_{k'}}{\partial x_k}(x) \end{bmatrix},$$

with $g_j\colon U \to \mathbb{R}$ the jth component of g.

Our goal will be to construct for each point m in a k-dimensional manifold M a *tangent space* T_mM, as well as for each smooth map $f\colon M \to N$ a *derivative* $d_mf\colon T_mM \to T_{f(m)}N$. The tangent space should satisfy the following properties:

(I) Each tangent space T_mM is a $\mathbb{R}$-vector space.

(II) In local coordinates it can be identified with $\mathbb{R}^k$ in a natural manner.

The derivative should satisfy similar properties:

(I') Each derivative d_mf is a linear map.

(II') In local coordinates it can be identified with the total derivative in a natural manner.

(III') It satisfies $d_m(\mathrm{id}_M) = \mathrm{id}_{T_mM}$, and the chain rule $d_m(g \circ f) = d_{f(m)}g \circ d_mf$.

We have not explained what "in a natural manner" means here. It is intended informally, but can be given some content by demanding that the identifications are compatible with changing coordinates.

There is a number of perspectives on tangent spaces and derivatives, leading to different but equivalent definitions. Which is most useful depends on your setting, and we will discuss five of them eventually. In the end, the "stating globally" part of our philosophy to state globally and prove locally, will allow us to dispense with the details of the definitions.

5.1.1 The algebraicists' definition

Intuitively, the tangent space to a k-dimensional submanifold of Euclidean space at some point is the k-dimensional affine linear subspace that best approximates it. However, we do not know (yet) that every smooth manifold is a submanifold of some Euclidean space, nor do we want to verify that the resulting definition is independent of the choice of such an embedding. So instead, we want a definition of T_mM that only refers to M and its maximal atlas. The first definition we will give, the algebraicists' one, does so, and will be our official one. However, you are free to use one of the definitions in the next section if those are more convenient for solving the problem at hand.

Germs of smooth maps and smooth functions

We start with the observation that the derivative of $f\colon M \to N$ at $m \in M$ should only depend on the behaviour of f in a small neighbourhood of m. Let us define an equivalence relation $\sim$ on the set

$$\{f\colon U \to N \mid U \subset M \text{ an open neighbourhood of } m,\ f \text{ smooth}\},$$

by saying that

$$f \sim g \text{ if there exists an open neighbourhood } V \text{ of } m \text{ such that } f|_V = g|_V.$$

Definition 5.1.1. The equivalence class of a smooth map $f\colon U \to N$ under $\sim$ is called *germ* of f at m, and denoted $\overline{f}\colon (M,m) \to N$. If we like to stress that $f(m) = n$, we will use the notation $\overline{f}\colon (M,m) \to (N,n)$.

We can compose germs: given $\overline{f}\colon (M,m) \to (N,n)$ and $\overline{g}\colon (N,n) \to (P,p)$, their composition is

$$\overline{g} \circ \overline{f} \coloneqq \overline{g \circ f},$$

leaving it to the reader to verify this is well-defined, i.e. independent of the choice of representatives.

Definition 5.1.2. A *function germ* is a germ $\overline{\alpha}\colon (M,m) \to \mathbb{R}$. The set of function germs is denoted $\mathcal{E}(M,m)$.

Pointwise addition, scaling, and multiplication of functions induces on $\mathcal{E}(M,m)$ the structure of an $\mathbb{R}$-algebra: this means it has addition, scaling, and multiplication operations

$$\overline{f} + \overline{g} := \overline{f+g}, \quad \lambda\overline{f} = \overline{\lambda f}, \quad \text{and} \quad \overline{f}\overline{g} := \overline{fg} \qquad \text{for } f,g \in \mathcal{E}(M,m) \text{ and } \lambda \in \mathbb{R},$$

These should satisfy appropriate commutativity, associativity, unitality, and distributivity axioms. We will leave it to the reader to verify these operations are well-defined, and satisfy these required properties (which will follow directly from the corresponding properties of the real numbers).

Example 5.1.3. Evaluation at $m \in M$ induces a function

$$\begin{aligned} \mathrm{ev}_m\colon \mathcal{E}(M,m) &\longrightarrow \mathbb{R} \\ \overline{f} &\longmapsto \overline{f}(m). \end{aligned}$$

This is an $\mathbb{R}$-*algebra homomorphism*, i.e. preserves addition, scaling, and multiplication.

We can precompose function germs in $\mathcal{E}(M,m)$ by a germ $f\colon (Q,q) \to (M,m)$, and thus get an $\mathbb{R}$-algebra homomorphism

$$\begin{aligned} f^*\colon \mathcal{E}(M,m) &\longrightarrow \mathcal{E}(Q,q) \\ \overline{\alpha} &\longmapsto \overline{\alpha} \circ f = \overline{\alpha \circ f}. \end{aligned}$$

The usual properties of composition of functions imply:

Lemma 5.1.4.

- f^* *is an* $\mathbb{R}$-*algebra homomorphism,*
- $\mathrm{id}^* = \mathrm{id}$*, and*
- $(g \circ f)^* = f^* \circ g^*$.

In particular, if ϕ is a diffeomorphism then ϕ^* is an isomorphism of $\mathbb{R}$-algebras; its inverse is given by $(\phi^{-1})^*$. Furthermore, since a germ only depend on maps on arbitrarily small open neighbourhoods of m, it suffices that ϕ is a local diffeomorphism.

We can apply this observation to a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ with $m \in V_\alpha$. By translation, we may assume without loss of generality that $\phi_\alpha(0) = m$. Then we can consider ϕ_α as a local diffeomorphism $U_\alpha \to M$ and hence it induces an isomorphism

$$(\phi_\alpha)^*\colon \mathcal{E}(M,m) \longrightarrow \mathcal{E}_k,$$

of $\mathcal{E}(M,m)$ with $\mathcal{E}_k := \mathcal{E}(\mathbb{R}^k, 0)$, the $\mathbb{R}$-algebra of functions germs $(\mathbb{R}^k, 0) \to \mathbb{R}$. Any two such identifications differ by an isomorphism $(\psi_{\beta\alpha})^*$ induced by a transition function.

From germs to the algebraicists' definition of tangent spaces

The idea behind the algebraicists' definition is that a vector $\vec{v}$ based at $m \in M$ induces a directional derivative of functions $f\colon M \to \mathbb{R}$, which we can imprecisely write as

$$f \longmapsto d_{\vec{v}}(f) := \frac{df(m+t\vec{v})}{dt}(0) \in \mathbb{R}. \tag{5.1}$$

(The difficulty is that we can not make sense of $m + tv$, but let us just go with it.) This only depends on the germ $\overline{f}$ of f at m. Furthermore, by the linearity of derivatives and the product rule, this should satisfy

$$d_{\vec{v}}(\overline{f}+\overline{g}) = d_{\vec{v}}(\overline{f}) + d_{\vec{v}}(\overline{g}), \quad d_{\vec{v}}(\lambda\overline{f}) = \lambda d_{\vec{v}}(\overline{f}),$$
$$\text{and} \quad d_{\vec{v}}(\overline{f}\overline{g}) = d_{\vec{v}}(\overline{f})\overline{g}(m) + \overline{f}(m)d_{\vec{v}}(\overline{g}).$$

Let us abstract this definition:

Definition 5.1.5. A *derivation* $X\colon \mathcal{E}(M,m) \to \mathbb{R}$ is a function which satisfies

- $X(\overline{f}+\overline{g}) = X(\overline{f}) + X(\overline{g})$,
- $X(\lambda\overline{f}) = \lambda X(\overline{f})$, and
- $X(\overline{f}\overline{g}) = X(\overline{f})\overline{g}(m) + \overline{f}(m)X(\overline{g})$.

Example 5.1.6. The value of X on the constant function 1 is given by

$$X(1) = X(1\cdot 1) = X(1)\cdot 1 + 1\cdot X(1) = 2\cdot X(1),$$

so $X(1) = 0$. As a consequence of linearity, $X(\text{constant function}) = 0$.

We can add and scale such derivations, making them into a $\mathbb{R}$-vector space:

$$(X+Y)(f) = X(f) + Y(f) \quad \text{and} \quad (\lambda X)(f) = \lambda X(f).$$

Definition 5.1.7. The *tangent space* $T_m M$ is the vector space $\mathrm{Der}(\mathcal{E}(M,m))$ of derivations $X\colon \mathcal{E}(M,m) \to \mathbb{R}$.

Let us recap: $\mathcal{E}(M,m)$ is the $\mathbb{R}$-algebra of germs at m of smooth functions $M \to \mathbb{R}$. We take derivations of this algebra, a notion inspired by directional derivatives. These form a vector space as in desideratum (I) but it remains to show that the vector space $T_m M$ is k-dimensional if M is k-dimensional as in desideratum (II). To do so, we use that the isomorphism $(\varphi_\alpha)^*\colon \mathcal{E}(M,m) \xrightarrow{\sim} \mathcal{E}_k = \mathcal{E}(\mathbb{R}^k, 0)$ induced by a chart induces a linear isomorphism

$$T_0\mathbb{R}^k = \mathrm{Der}(\mathcal{E}_k) \xrightarrow{\sim} \mathrm{Der}(\mathcal{E}(M,m)) = T_m M.$$

Thus it suffices to prove that $T_0\mathbb{R}^k$ is k-dimensional. Unlike on M, on $\mathbb{R}^k$ we *can* make sense of addition, and hence the directional derivatives of (5.1) with respect to each of the k coordinate directions give derivations

$$\frac{\partial}{\partial x_i}\colon \mathcal{E}_k \longrightarrow \mathbb{R}$$
$$\overline{f} \longmapsto \frac{\partial f}{\partial x_i}(0).$$

To see that these are linearly independent, apply them to the coordinate functions $x_j\colon (x_1,\ldots,x_k) \mapsto x_j$. Every other derivation is a linear combination of these:

Proposition 5.1.8. *The derivations $\frac{\partial}{\partial x_i}$ form a basis of $T_0\mathbb{R}^k$, and in particular the latter is k-dimensional.*

We will use the following lemma:

Lemma 5.1.9. *Let $U \subset \mathbb{R}^k$ be an open neighborhood and $f\colon U \to \mathbb{R}$ a smooth function. Then there exist smooth functions $f_1, \ldots, f_k\colon U \to \mathbb{R}$ such that*

$$f(x) = f(0) + \sum_{i=1}^{k} x_i f_i(x).$$

Proof. The fundamental theorem of analysis implies

$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_k)dt = \sum_{i=1}^{n} x_i \int_0^1 d_i f(tx_1, \ldots, tx_n)dt,$$

with $d_i f$ the partial derivative in the ith coordinate direction. So we have that

$$f_i(x) = \int_0^1 d_i f(tx_1, \ldots, tx_n)dt. \qquad \square$$

This implies that for germs we have $\overline{f} = f(0) + \sum_i \overline{x}_i \overline{f_i}$.

Proof of Proposition 5.1.8. We prove that $X = \sum_{i=1}^{k} X(x_i)\frac{\partial}{\partial x_i}$ by proving that

$$Y := X - \sum_{i=1}^{k} X(x_i)\frac{\partial}{\partial x_i}$$

vanishes on all germs. By construction, it vanishes on the coordinate function. Then we have that

$$\begin{aligned} Y(\overline{f}) &= Y\left(f(0) + \overline{x}_i\overline{f_i}\right) \\ &= Y(f(0)) + \sum_i Y(\overline{x}_i\overline{f_i}) \\ &= \sum_i Y(\overline{x}_i)\overline{f_i}(0) \\ &= 0. \end{aligned}$$

Here we use that $\overline{x}_i$ evaluates to 0 at the origin, and that $Y(\overline{x}_i)$ vanishes by construction. $\square$

The algebraicists' definition of derivatives

A smooth map $f\colon M \to N$ sending m to n induces a map of germs $f^*\colon \mathcal{E}(N, n) \to \mathcal{E}(M, m)$, which in turn induces a map of tangent spaces

$$\begin{aligned} d_m f\colon T_m M &\longrightarrow T_n N \\ X &\longmapsto X \circ f^*. \end{aligned}$$

This is the *derivative of f at m*. From the properties of f^*, we easily deduce the basic properties of the derivative, desiderata (I') and (III'):

Lemma 5.1.10.

(i) $d_m f$ *is a linear map*

(ii) $d_m \mathrm{id} = \mathrm{id}$, *and*

(iii) $d_m(g \circ f) = d_{f(m)} g \circ d_m f$.

You may recognize (iii) as an incarnation of the chain rule. We will compare it to the chain rule in multivariable calculus later in this section.

Example 5.1.11. If $f\colon M \to N$ is a diffeomorphism, then it follows from (ii) and (iii) that $d_m f$ is invertible with inverse $d_{f(m)} f^{-1}$.

Example 5.1.12. To compute the derivative, you can often exploit the chain rule. Recall that $\pi_1\colon X \times Y \to X$ has derivative

$$d_{(x,y)}\pi_1\colon T_x X \oplus T_y Y = T_{(x,y)}(X \times Y) \longrightarrow T_x X,$$

is given by projection onto the first summand. The analogous statement is true for $\pi_2\colon X \times Y \to Y$.

We will deduce from this that the diagonal map

$$\begin{aligned} \Delta\colon M &\longrightarrow M \times M \\ m &\longmapsto (m,m) \end{aligned}$$

has derivative $T_m\Delta\colon T_m M \to T_{m\times m}(M \times M) = T_m M \oplus T_m M$ given by $v \mapsto (v,v)$. To see this, observe that $\pi_1 \circ \Delta$ and $\pi_2 \circ \Delta$ have derivatives given by the first and second components of $T_m\Delta$. We apply the chain rule to $\pi_1 \circ \Delta = \mathrm{id}_M = \pi_2 \circ \Delta$. For example, for the first equality: the first component of $T_m\Delta(v)$ is given by

$$T_{(m,m)}\pi_1 \circ T_m\Delta(v) = T_m(\pi_1 \circ \Delta)(v) = T_m(\mathrm{id}_M)(v) = v.$$

For example, this implies that the diagonal map has injective derivative everywhere.

Let us finally describe explicitly $T_m f$ in terms of charts, and verify desideratum (II'). Fix a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ of M such that $\phi_\alpha(0) = m$, and a chart $(U'_{\alpha'}, V'_{\alpha'}, \phi'_{\alpha'})$ of N such that $\phi'_{\alpha'}(0) = f(m)$. Let us denote $f(m)$ by n. What is the dashed linear map which makes the following diagram commute?

$$\begin{tikzcd} T_m M \arrow[r, "d_m f"] & T_n N \\ \mathbb{R}^k \cong T_0\mathbb{R}^k \arrow[u, "\cong"', "d_0\phi_\alpha"] \arrow[r, dashed] & \mathbb{R}^{k'} \cong T_0\mathbb{R}^{k'}. \arrow[u, "\cong"', "d_0\phi'_{\alpha'}"] \end{tikzcd} \tag{5.2}$$

Lemma 5.1.13. *It is the total derivative* $D_0((\phi'_{\alpha'})^{-1} \circ f \circ \phi_\alpha)$.

Proof. As $(d_0\phi'_{\alpha'})^{-1} = d_n((\phi'_{\alpha'})^{-1})$ by Example 5.1.11, and

$$d_n(\phi'_{\alpha'})^{-1} \circ d_m f \circ d_0\phi_\alpha = d_0((\phi'_{\alpha'})^{-1} \circ f \circ \phi_\alpha)$$

by (iii), it suffices to compute explicitly the derivative of $g\colon \mathbb{R}^k \to \mathbb{R}^{k'}$ at the origin; we will substitute $g = (\phi'_{\alpha'})^{-1} \circ f \circ \phi_\alpha$. We write $g_j\colon \mathbb{R}^k \to \mathbb{R}$ for the jth component of g, $1 \leqslant j \leqslant k'$.

Given $\overline{h} \in \mathcal{E}_{k'}$, we can use the chain rule to compute that

$$d_0 g(\tfrac{\partial}{\partial x_i})(\overline{h}) = \frac{\partial(\overline{h} \circ g)}{\partial x_i} = \sum_{j=1}^{k'} \frac{\partial \overline{h}}{\partial y_j}(0)\frac{\partial g_j}{\partial x_i}(0) = \sum_{j=1}^{k'} \frac{\partial g_j}{\partial x_i}(0)\tfrac{\partial}{\partial y_j}(\overline{h}).$$

As this is true for all $\overline{h}$, the $\frac{\partial}{\partial y_j}$-component is $\frac{\partial g_j}{\partial x_i}(0)$. These are exactly the entries of the total derivative matrix. □

Remark 5.1.14. We can use this to justify calling $d_m(g \circ f) = d_{f(m)} g \circ d_m f$ a chain rule, by proving that under charts it reduces to the chain rule you already know. Fixing a third chart, we have a triple of commutative diagrams (three instances of (5.2))

$$\begin{tikzcd} T_m M \arrow[rr, "d_m f"] & & T_n N \\ \mathbb{R}^k \cong T_0\mathbb{R}^k \arrow[u, "\cong"', "T_0\phi_\alpha"] \arrow[rr, "D_0((\phi'_{\alpha'})^{-1}\circ f\circ\phi_\alpha)"] & & \mathbb{R}^{k'} \cong T_0\mathbb{R}^{k'}, \arrow[u, "\cong"', "T_0\phi'_{\alpha'}"] \end{tikzcd}$$

$$\begin{tikzcd} T_{f(m)} N \arrow[rr, "d_{f(m)} g"] & & T_{g\circ f(m)} P \\ \mathbb{R}^{k'} \cong T_0\mathbb{R}^{k'} \arrow[u, "\cong"', "T_0\phi'_{\alpha'}"] \arrow[rr, "D_0((\phi''_{\alpha''})^{-1}\circ g\circ\phi'_{\alpha'})"] & & \mathbb{R}^{k''} \cong T_0\mathbb{R}^{k''}, \arrow[u, "\cong"', "T_0\phi''_{\alpha''}"] \end{tikzcd}$$

$$\begin{tikzcd} T_m N \arrow[rr, "d_m(g\circ f)"] & & T_{g\circ f(m)} P \\ \mathbb{R}^k \cong T_0\mathbb{R}^k \arrow[u, "\cong"', "T_0\phi_\alpha"] \arrow[rr, "D_0((\phi''_{\alpha''})^{-1}\circ g\circ f\circ\phi_\alpha)"] & & \mathbb{R}^{k''} \cong T_0\mathbb{R}^{k''}. \arrow[u, "\cong"', "T_0\phi''_{\alpha''}"] \end{tikzcd}$$

Identifying the term $d_{f(m)} g \circ d_m f$ in charts using the vertical arrows in the three commutative diagrams pictured above, we get a composition of total derivatives

$$D_0((\phi''_{\alpha''})^{-1} \circ g \circ \phi'_{\alpha'}) \circ D_0((\phi'_{\alpha'})^{-1} \circ f \circ \phi_\alpha).$$

By the ordinary chain rule this is the total derivative

$$D_0((\phi''_{\alpha''})^{-1} \circ g \circ \phi'_{\alpha'} \circ (\phi'_{\alpha'})^{-1} \circ f \circ \phi_\alpha) = D_0((\phi''_{\alpha''})^{-1} \circ g \circ f \circ \phi_\alpha),$$

which is indeed $d_m(g \circ f)$ under the above identification.

Thus, we can combine the three squares into a larger commutative diagram combining the general chain rule and the chain rule in local coordinates:

$$\begin{tikzcd}
T_mM \arrow[r, "d_m f"] \arrow[rr, bend left, "d_m(g\circ f)"] & T_nN \arrow[r, "D_{f(m)}g"] & d_{g\circ f(m)}P \\
\mathbb{R}^k \cong T_0\mathbb{R}^k \arrow[u, "T_0\phi_\alpha"', "\cong"] \arrow[r, "D_0((\phi'_{\alpha'})^{-1}\circ f\circ\phi_\alpha)"] \arrow[rr, bend right, "D_0((\phi''_{\alpha''})^{-1}\circ g\circ f\circ\phi_\alpha)"'] & \mathbb{R}^{k'} \cong T_0\mathbb{R}^{k'} \arrow[u, "T_0\phi'_{\alpha'}"', "\cong"] \arrow[r, "D_0((\phi''_{\alpha''})^{-1}\circ f\circ\phi'_{\alpha'})"] & \mathbb{R}^{k''} \cong T_0\mathbb{R}^{k''} \arrow[u, "T_0\phi''_{\alpha''}"', "\cong"]
\end{tikzcd}$$

5.2 Alternative definitions of tangent spaces and derivatives

Recall that we are giving five definitions of the tangent space T_mM, and have just given the first. In this section we give three other definitions, leaving a final one to the Problem 10.

5.2.1 The definition for submanifolds of Euclidean space

You probably have an intuition for the tangent space at m to some k-dimensional smooth submanifold $M \subset \mathbb{R}^n$. Informally, it is the k-dimensional affine plane in $\mathbb{R}^n$ through $m \in M$, which is the best linear approximation to M. Before making this precise, we give an example:

Example 5.2.1. By definition, a point $x \in S^k \subset \mathbb{R}^{k+1}$ is given by a unit length vector in $\mathbb{R}^{k+1}$. Then the tangent space T_xS^k is the k-dimensional affine plane given by

$$T_xS^n = \{x + v \mid v \perp x\}.$$

Note that, upon translating m back to the origin, this affine plane yields a linear subspace of $\mathbb{R}^n$. This gives T_mM the structure of an m-dimensional real vector space.

To define T_mM rigorously, we fix a charts $(U_\alpha, V_\alpha, \phi_\alpha)$ of M such that $m \in V_\alpha$, and let $x = \phi_\alpha^{-1}(m)$. Then from the inclusion $i\colon M \to \mathbb{R}^n$, we can construct a smooth map between open subsets of Euclidean space

$$i \circ \phi_\alpha \colon \mathbb{R}^k \supset U_\alpha \longrightarrow \mathbb{R}^n.$$

The best linear approximation to this smooth map at x is given in terms of the total derivative $D_x(i \circ \phi_\alpha)$ as

$$\mathbb{R}^k \supset U_\alpha \ni y \longmapsto (i \circ \phi_\alpha)(x) + D_x(i \circ \phi_\alpha)(y - x) \in \mathbb{R}^n.$$

It is a consequence of the definition of a submanifold that $D(i\circ\phi_\alpha)_x$ is an injective linear map; indeed, in terms of some other chart of $\mathbb{R}^n$ it is a restriction of the inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$. This tells us that:

Definition 5.2.2. One definition of the tangent space T_mM is as

$$T_m^{\text{submfd}}M := m + D_x(i \circ \phi_\alpha)(\mathbb{R}^k),$$

considered as a k-dimensional real vector space by its identification with $D_x(i \circ \phi_\alpha)(\mathbb{R}^k)$.

We need to verify that this is independent of the choice of chart. This is the case because if we use another chart $(U_\beta, V_\beta, \phi_\beta)$, we have

$$i \circ \phi_\beta = (i \circ \phi_\alpha) \circ (\phi_\alpha^{-1} \circ \phi_\beta) = (i \circ \phi_\alpha) \circ \psi_{\beta\alpha},$$

so its total derivative is given by $D_x(i \circ \phi_\alpha) \circ D_{x'}(\psi_{\beta\alpha})$. Since $\psi_{\beta\alpha}$ is a diffeomorphism, $D_{x'}(\psi_{\beta\alpha})$ is a linear isomorphism and hence

$$D_{x'}(i \circ \phi_\beta)(\mathbb{R}^k) = D_x(i \circ \phi_\alpha)(\mathbb{R}^k).$$

Relation to algebraicists' definition

We have previously identified $T_0\mathbb{R}^n$ with the n-dimensional real vector space spanned by the derivations $\partial/\partial x_i$. You can think of this as applying the formalism above to $M = \mathbb{R}^n$, using the standard chart $(\mathbb{R}^n, \mathbb{R}^n, \mathrm{id})$.

Given an inclusion $i\colon M \to \mathbb{R}^n$, where without loss of generality we may assume by translation that $i(m) = 0$, we can compute the derivative of i at m with respect to the standard chart of $\mathbb{R}^n$ and some chart $(U_\alpha, V_\alpha, \phi_\alpha)$ of M with $\phi_\alpha(0) = m$. By the chain rule there is a commutative diagram of linear maps

$$\begin{tikzcd} T_mM \arrow[r, "d_mi"] & T_0\mathbb{R}^n \\ \mathbb{R}^k \cong T_0\mathbb{R}^k \arrow[u, "\cong"', "T_0\phi_\alpha"] \arrow[r, "D_0(i\circ\phi_\alpha)"] & \mathbb{R}^n \cong T_0\mathbb{R}^n. \arrow[u, "\cong"', "\mathrm{id}"] \end{tikzcd}$$

Because M is a submanifold d_mi is injective, as in terms of appropriate charts it is the derivative of the inclusion $\mathbb{R}^k \to \mathbb{R}^n$. This tells us that we could have defined T_mM as the image of the linear map d_mi. By the commutative diagram, this linear subspace is the same as the image of the total derivative $D_0(i \circ \phi_\alpha)$. Undoing the translation of $i(m)$ to the origin, we recover the $T_m^{\text{submfd}}M$. We conclude that there is a preferred linear isomorphism

$$T_mM \xrightarrow{\cong} T_m^{\text{submfd}}M.$$

5.2.2 The physicists' definition

For physicists, a tangent vector is described in terms of a chart (thought of as a local coordinate system), which transforms in a certain way when passing to other local coordinates. That is, an element of T_mM is an equivalence class of a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ such that $\phi_\alpha(0) = M$ and a vector $v \in \mathbb{R}^k$. The equivalence relation tells us that v transforms as expected: by applying the total derivative of the transition function $\psi_{\beta\alpha}$.

Definition 5.2.3. The physicists' definition of the *tangent space of M at m* is

$$T_m^{\mathrm{phys}} M = \left(\bigsqcup_{(U_\alpha, V_\alpha, \phi_\alpha)} \mathbb{R}^k \right) / \simeq$$

where the disjoint union is over all charts with $m = \phi_\alpha(0)$ and the equivalence relation $\simeq$ is given by

$$(\alpha, \vec{v}) \simeq (\beta, \vec{w}) \quad \text{if and only if} \quad \vec{w} = D_0 \psi_{\beta\alpha}(\vec{v}).$$

Remark 5.2.4. This definition reflects the experimental roots of physical theories: the transformation rule under change of local coordinates for physical quantities is determined experimentally, and a mathematical framework is built on top of these results.

Since each $D_0\psi_{\beta\alpha}$ is a linear map, addition and scalar multiplication in each copy of $\mathbb{R}^k$ induce a vector space structure on $T_m^{\mathrm{phys}} M$. Since each copy of $\mathbb{R}^k$ is identified with every other copy, this is a k-dimensional vector space. To get a corresponding notion of derivative, we observe that any smooth map $f \colon M \to N$ induces a map

$$\begin{aligned} d_f^{\mathrm{phys}} \colon T_m^{\mathrm{phys}} M &\longrightarrow T_{f(m)}^{\mathrm{phys}} N \\ [\alpha, \vec{v}] &\longmapsto \left[\alpha', D_0((\phi'_{\alpha'})^{-1} \circ f \circ \phi_\alpha)(\vec{v})\right]. \end{aligned}$$

Relation to algebraicists' definition

The maps

$$\begin{aligned} \mathbb{R}^k &\longrightarrow T_m M \\ (\alpha, \vec{v}) &\longmapsto (D_0\phi_\alpha)\Big(\sum_i v_i \partial/\partial x_i\Big) \end{aligned}$$

are compatible with the equivalence relation, and thus induce a linear map $T_m^{\mathrm{phys}}(M) \to T_m M$. On representatives of the form $(\alpha, \vec{v})$, its composition with the linear isomorphism $(D_0\phi_\alpha)^{-1} \colon T_m M \to \mathbb{R}^k$ is given by $(\alpha, \vec{v}) \mapsto \vec{v}$, so this is an isomorphism. We conclude that there is a preferred linear isomorphism

$$T_m^{\mathrm{phys}}(M) \xrightarrow[\textcircled{1}]{\cong} T_m M.$$

This identification is compatible with the construction of derivatives: we leave it to the reader to verify that the following diagram of linear maps commutes

$$\begin{tikzcd} T_m^{\mathrm{phys}} M \arrow[r, "d_m^{\mathrm{phys}} f"] \arrow[d, "\cong"', "\textcircled{1}"] & T_{f(m)}^{\mathrm{phys}} N \arrow[d, "\cong"', "\textcircled{1}"] \\ T_m M \arrow[r, "d_m f"] & T_{f(m)} N. \end{tikzcd}$$

5.2.3 The geometers' definition

For geometers, a tangent vector is the derivative of a curve. As such, it is an equivalence class of germs of smooth maps

$$\overline{\gamma}\colon (\mathbb{R}, 0) \longrightarrow (M, m).$$

Because we want to avoid a circular definition, we can not (yet) refer to the derivative of this map. However, we can take a function germ $\overline{g}\colon (M, m) \to \mathbb{R}$ and compute

$$\tfrac{d}{dt} g \circ \gamma(0),$$

a derivative of a real-valued function on a neighbourhood of the origin in $\mathbb{R}$. This allows us to introduce an relation $\approx$ on curves through m, given by

$$\overline{\gamma} \approx \overline{\eta} \quad \text{if and only if} \quad \tfrac{d}{dt} g \circ \gamma(0) = \tfrac{d}{dt} g \circ \eta(0) \text{ for all } g\colon (M, m) \to \mathbb{R}.$$

That is, if γ and η define the same directional derivative.

Definition 5.2.5. The geometers' definition of the *tangent space of M at m* is

$$T_m^{\text{geom}} M := \{\text{germs } (\mathbb{R}, 0) \to (M, m)\}/\!\approx.$$

We will explain how to make it a vector space momentarily. To get a corresponding notion of derivative, we observe that any smooth map $f\colon M \to N$ induces a map

$$\begin{aligned} d_f^{\text{geom}}\colon T_m^{\text{geom}} M &\longrightarrow T_{f(m)}^{\text{geom}} N \\ [\overline{\gamma}] &\longmapsto [\overline{f \circ \gamma}]. \end{aligned}$$

Relation to algebraicists' definition

There is a map

$$\begin{aligned} T_m^{\text{geom}} M = \{\text{germs } (\mathbb{R}, 0) \to (M, m)\}/\!\approx &\longrightarrow T_m M = \operatorname{Der}(\mathcal{E}(M, m)) \\ [\overline{\gamma}] &\longmapsto \left(\overline{h} \mapsto \frac{d(h \circ \gamma)}{dt}\right) \end{aligned}$$

By evaluation on coordinate functions in a chart, this is seen to be injective. By construction of curves in the same chart, this is seen to be surjective. Hence it is a bijection. In particular, we can use this to make $T_m^{\text{geom}} M$ into a vector space, getting tautologically a linear isomorphism.

$$T_m^{\text{geom}} M \xrightarrow[②]{\cong} T_m M.$$

Again, this is compatible with the construction of derivatives: we leave it to the reader to verify that the following diagram of linear maps commutes

$$\begin{tikzcd} T_m^{\text{geom}} M \arrow[r, "d_m^{\text{geom}} f"] \arrow[d, "\cong"', "②"] & T_{f(m)}^{\text{geom}} N \arrow[d, "\cong"', "②"] \\ T_m M \arrow[r, "d_m f"] & T_{f(n)} N. \end{tikzcd}$$

5.3 Problems

Problem 10 (The algebraic geometers' definition)**.** In this problem you will give the algebraic geometers' definition of a tangent space.

(a) Prove that there is a unique maximal ideal of $\mathcal{E}(M, m)$, given by $\mathfrak{m}_m = \{f \mid f(m) = 0\}$.

(b) Prove that for $(M, m) = (\mathbb{R}^k, 0)$, the maximal ideal $\mathfrak{m}_0$ is spanned by the coordinate functions $x_1, \dots, x_k$.

(c) Prove that $\mathcal{E}(M, m)/\mathfrak{m}_m$ is a 1-dimensional $\mathbb{R}$-vector space, and $\mathfrak{m}_m/\mathfrak{m}_m^2$ is k-dimensional if M is k-dimensional.

The algebraic geometers' definition of the tangent space of M at m is

$$T_m^{\mathrm{ag}} M := (\mathfrak{m}_m/\mathfrak{m}_m^2)^*.$$

(d) Construct a linear map $d_m^{\mathrm{ag}} f \colon T_m^{\mathrm{ag}} M \to T_{f(m)}^{\mathrm{ag}} N$ for each smooth map $f \colon M \to N$. Prove it satisfies $d_m^{\mathrm{ag}} \mathrm{id} = \mathrm{id}$ and $d_m^{\mathrm{ag}}(g \circ f) = d_{f(m)}^{\mathrm{ag}} g \circ d_m^{\mathrm{ag}} f$.

(e) Construct a linear map $T_m M \to T_m^{\mathrm{ag}} M$ and prove it is an isomorphism.

Chapter 6

Tangent bundles

We now assemble the tangent spaces to tangent bundles, and the derivatives of a smooth map to a map of tangent bundles. This appears in Chapter 2 of [BJ82]. See also [Tu11, Chapters 6, 8].

6.1 Vector bundles

Recall that the total derivative of a smooth map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ at a point $x \in \mathbb{R}^m$ is a linear map $\mathbb{R}^m \to \mathbb{R}^n$ which with respect to the standard coordinates is given by the $(n \times m)$-matrix of partial derivatives of its components at x. Importantly it depends smoothly on x. On smooth manifolds the domains and targets will depend on points m and $f(m)$ respectively, so we can not state the smooth dependence on m without first assembling the tangent spaces and derivatives to each $m \in M$ together in an appropriate object, known as a *vector bundle*. This is one of the other geometric objects studied by differential topology, in addition to smooth manifolds, and the tangent bundle is the prototypical example.

6.1.1 Vector bundles

We start with the topological variant, before adding in the smooth structure later in this lecture:

Definition 6.1.1. A *k-dimensional vector bundle over a topological space X* is a topology on the disjoint union $E = \bigsqcup_{x\in X} E_x$ of a collection of real vector spaces, such that

(i) the function $p\colon E \to X$ sending E_x to x is continuous,

(ii) for each $x \in X$ there exists an open subset $V \subset X$ containing x and a homeomorphism
$$\zeta\colon \bigsqcup_{x\in V} E_x \xrightarrow{\cong} V \times \mathbb{R}^k$$
that restricts to an invertible linear map $E_x \to \{x\} \times \mathbb{R}^k$ for each $x \in V$.

The continuous map p is called the *projection*, E the *total base*, X the *base*, and each E_x a *fibre*. Finally, the pair (V, ζ) is called a *bundle chart*.

Example 6.1.2. The cartesian product $X \times \mathbb{R}^k$ has an evident structure of a k-dimensional vector bundle. We call this the *trivial k-dimensional vector bundle over X*. The property in Definition 6.1.1 is often referred to as a *local triviality* condition, as it is says that E locally looks like such a trivial bundle.

Example 6.1.3. The real projective space $\mathbb{R}P^n$ is the space of lines in $\mathbb{R}^{n+1}$. There is a 1-dimensional vector bundle over it with fibre of L given by those $v \in \mathbb{R}^{n+1}$ which lie in L. This is the *canonical bundle*. More precisely, writing $\mathbb{R}P^n = S^n/\{\pm 1\}$, we have

$$E_{[x]} = \{v \in \mathbb{R}^{n+1} \mid v = \lambda x \text{ for some } \lambda \in \mathbb{R}\} \qquad \text{for } [x] \in \mathbb{R}P^n.$$

We topologise $\bigsqcup_{[x]} E_{[x]}$ as a subspace of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$. The local triviality condition is verified using charts.

6.1.2 Maps between vector bundles

Definition 6.1.4. Let $p\colon E \to X$ and $p'\colon E' \to X'$ be vector bundles (possibly of different dimension). For a continuous map $F\colon E \to E'$ to be a *map of vector bundles*, the first requirement is that there is a continuous map $f\colon X \to X'$ the following diagram commute

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"] & X'. \end{tikzcd}$$

Then F restricts to a map of fibres $F_x\colon E_x \to E'_{f(x)}$, and the second requirement is that this is a linear map.

Note that f is uniquely determined by F, and we say that F *covers* f or F *is over* f. It is clear that the identity is a map of vector bundles, and that maps of vector bundles are closed under composition.

Definition 6.1.5. An *isomorphism of vector bundles* is a map of vector bundles which admit an inverse map of vector bundles.

Example 6.1.6. Over S^1 we have exactly two 1-dimensional vector bundles up to isomorphism: the trivial one and the “Möbius strip” bundle. The latter is given by the canonical bundle over $\mathbb{R}P^1 \cong S^1$, and can be concretely given by taking $[0,1] \times \mathbb{R}$ and identifying the endpoints by $(0, v) \sim (1, -v)$.

Example 6.1.7. Let $X \times \mathbb{R}^m$ be a trivial bundle. The $(m \times m)$-matrices $\mathrm{Mat}_m(\mathbb{R})$ are topologised by identifying them with $\mathbb{R}^{m^2}$ through their entries. Then any continuous map $A\colon X \to \mathrm{Mat}_m(\mathbb{R})$ gives rise to a map of vector bundles

$$\begin{aligned} X \times \mathbb{R}^m &\longrightarrow X \times \mathbb{R}^m \\ (x, v) &\longmapsto (x, A(x)(v)). \end{aligned}$$

This is an isomorphism of vector bundles if and only if A takes values in $\mathrm{GL}_m(\mathbb{R}) \subset \mathrm{Mat}_m(\mathbb{R})$, the subset of invertible matrices.

6.1.3 Smooth vector bundles

As for topological manifolds, we can package the data of k-dimensional vector bundle over a topological space into an atlas: the collection of bundle charts (V, ζ) for (p, E, X) with V covering X is called a *bundle atlas*. As in the case of smooth atlases, we can define maximal bundle atlases and prove that every bundle atlas is contained in a unique maximal bundle atlas.

A bundle atlas has transition functions: taking (V_α, ζ_α) and (V_β, ζ_β), the composition

$$(V_\alpha \cap V_\beta) \times \mathbb{R}^k \xrightarrow{\zeta_\alpha^{-1}} p^{-1}(V_\alpha \cap V_\beta) \xrightarrow{\zeta_\beta} (V_\alpha \cap V_\beta) \times \mathbb{R}^k$$

is necessary of the form $(x, v) \mapsto (x, \xi_{\alpha\beta}(x)(v))$ for a linear map $\xi_{\alpha\beta}(x)\colon \mathbb{R}^k \to \mathbb{R}^k$ depending continuously on $x \in V_\alpha \cap V_\beta \subseteq X$.

If the base is a smooth manifold, so are the open subsets $V_\alpha \cap V_\beta$. Recall that $\mathrm{GL}_k(\mathbb{R})$ is an open subset of $\mathbb{R}^{k^2}$ and hence inherits a smooth structure, we can make sense of whether these transition functions are smooth. A bundle atlas is smooth if all transition functors are smooth.

Definition 6.1.8. Suppose M is a smooth manifold. Then a *smooth vector bundle*(p, E, M) is a vector bundle with a maximal smooth bundle atlas.

The proof of the following is left as a problem:

Lemma 6.1.9. *If (p, E, M) is a smooth vector bundle then there is a unique maximal atlas on E such that all bundle charts $\zeta_\alpha\colon p^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^k$ are diffeomorphisms and $p\colon E \to M$ is a smooth map.*

When we have a pair of vector bundles (p, E, M) and (p', E', M') and a map $F\colon E \to E'$ of vector bundles over $f\colon M \to M'$, then we can use the bundle charts to write

$$(V_\alpha \cap f^{-1}(V'_{\alpha'})) \times \mathbb{R}^k \xrightarrow{\zeta_\alpha^{-1}} p^{-1}(V_\alpha \cap f^{-1}(V'_{\alpha'})) \xrightarrow{F} (p')^{-1}(V'_{\alpha'}) \xrightarrow{\zeta'_{\alpha'}} V'_{\alpha'} \times \mathbb{R}^{k'}.$$

As before, this preserves the first coordinate and hence is encoded by a continuous map $(V_\alpha \cap f^{-1}(V'_{\alpha'})) \to \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{k'})$. We can ask this to be smooth, and if the vector bundles are smooth this is independent of the choice of bundle charts. If all these maps are smooth, we say that the map $F\colon (p, E, M) \to (p', E', M')$ of vector bundles is *smooth*. This is in particular always a smooth map between the manifolds M and M'.

6.2 The tangent bundle and the derivative

In the previous lecture, we described how to assign a vector space $T_m M$ to each $m \in M$, as well as maps

$$d_m f\colon T_m M \longrightarrow T_{f(m)} N,$$

which satisfy the desiderata:

(I') $d_m f$ is a linear map.

(II') In local coordinates $T_m M$ is $\mathbb{R}^k$ and $d_m f$ is the total derivative.

(III') $d_m \mathrm{id} = \mathrm{id}$ and $d_m(g \circ f) = d_{f(m)} g \circ d_m f$.

We next explain how to patch together the vector spaces $T_m M$ to a smooth vector bundle TM over M, and the linear maps $d_m f$ to a map $df\colon TM \to TN$ of smooth vector bundle for each smooth map $f\colon M \to N$. These should satisfy analogous desiderata:

(I") df is a map of vector bundles.

(II") In local coordinates TM is given by $\mathbb{R}^k$'s and df by the total derivatives.

(III") $d(\mathrm{id}) = \mathrm{id}$ and $d(g \circ f) = dg \circ df$,

6.2.1 Constructing the tangent bundle

To construct the tangent bundle TM of a manifold, we shall employ a general construction, presenting a vector bundle as a colimit of trivial bundles.

Definition 6.2.1. A *k-dimensional pre-vector bundle* over a space X is a disjoint union $E = \bigsqcup_{x \in X} E_x$ of a collection of real vector space E_x, together with a collection $\mathcal{B} = \{(V_\alpha, \zeta_\alpha)\}$ of open subsets V_α that cover X and bijections

$$\zeta_\alpha\colon \bigsqcup_{x \in V_\alpha} E_x \xrightarrow{\cong} V_\alpha \times \mathbb{R}^k$$

that restrict to invertible linear maps $E_x \to \{x\} \times \mathbb{R}^k$. Furthermore, we require that all transition functions $\xi_{\alpha\beta}\colon V_\alpha \cap V_\beta \to \mathrm{GL}_k(\mathbb{R})$ are continuous.

That is, a pre-vector bundle is essentially a vector bundle that is of yet without a topology on its total space. However, we have:

Lemma 6.2.2. *There is a unique topology on E so that $\mathcal{B} = \{(V_\alpha, \zeta_\alpha)\}$ is a bundle atlas for (p, E, B).*

Proof sketch. Give E the finest topology such that all ζ_α are continuous. □

If we replace X by a manifold M, we can similarly define k-dimensional smooth vector bundles, by demanding that all $\xi_{\alpha\beta}$ are smooth. Using the above construction then makes (p, E, M) into a smooth vector bundle. In particular, we can define the tangent bundle TM by prescribing a smooth pre-vector bundle on M:

- $TM = \bigsqcup_{m \in M} T_m M$,
- $\mathcal{B} = \{V_\alpha, \zeta_\alpha\}$ where $(U_\alpha, V_\alpha, \phi_\alpha)$ is ranges over the charts of the maximal atlas of M, and
- $\zeta_\alpha\colon \bigsqcup_{m \in M} T_m M \to V_\alpha \times \mathbb{R}^k$ is given by

$$(m, v) \longmapsto \left(m, (d_{\phi_\alpha^{-1}(m)} \phi_\alpha)^{-1}(v)\right).$$

Implicitly, we are using here the identifications of $T_{\phi_\alpha^{-1}(m)} \mathbb{R}^k$ with $\mathbb{R}^k$ using the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$.

Definition 6.2.3. The smooth vector bundle TM over M constructed from this pre-vector bundle is the *tangent bundle* to M.

Example 6.2.4. If $U \subset \mathbb{R}^k$ is open, then $TU = U \times \mathbb{R}^k$.

Note that this does not depend on the exact construction of the tangent spaces $T_m M$, but only that it satisfies the desiderata.

The tangent bundle is itself a smooth manifold. Indeed, there is a unique $2k$-dimensional maximal smooth atlas on TM such that each of the local trivializations $TM|_U \cong U \times \mathbb{R}^k$ induced by a chart of M is a diffeomorphism. As a consequence, the projection map $TM \to M$ is a smooth map, as is the 0-*section* $s_0 \colon M \to TM$; its image is a k-dimensional submanifold of TM diffeomorphic to M.

6.2.2 The derivative and its properties

It is now easy to define the *derivative* $df \colon TM \to TN$ of a smooth map $f \colon M \to N$. This will be a map of vector bundles which covers f, and hence it suffices to give linear maps $d_m f \colon T_m M \to T_{f(n)} N$ and verify that these are continuous and in fact smooth. Of course, we will take these linear maps the derivatives as we defined before.

Lemma 6.2.5. *The derivatives $d_m f \colon T_m M \to T_{f(n)} N$ assemble to a smooth bundle map $df \colon TM \to TN$.*

Proof. Since being smooth is a local property, it suffices to check this with respect to the bundle charts defining TM and TN, i.e. those arising from charts. That is, we need to prove that

$$\begin{aligned}(V_\alpha \cap f^{-1}(V'_\beta)) \times \mathbb{R}^k &\longrightarrow V'_\beta \times \mathbb{R}^{k'} \\ (m, v) &\longmapsto \Big(f(m), [(d_{(\phi'_\beta)^{-1}(f(m))} \phi'_\beta)^{-1} \circ d_m f \circ d_{(\phi'_\beta)^{-1}(m)} \phi'_\beta(m)](v)\Big)\end{aligned}$$

is smooth. To do so, we precompose it with the diffeomorphism

$$\phi_\alpha \times \mathrm{id} \colon \phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_\beta)) \times \mathbb{R}^k \xrightarrow{\cong} (V_\alpha \cap f^{-1}(V'_\beta)) \times \mathbb{R}^k$$

and postcompose it with the inverse of

$$\phi_\beta \times \mathrm{id} \colon U'_\beta \times \mathbb{R}^k \xrightarrow{\cong} V'_\beta \times \mathbb{R}^k.$$

The result is the map $\phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_\beta)) \times \mathbb{R}^k \longrightarrow U'_\beta \times \mathbb{R}^{k'}$ between trivial vector bundles over open subsets of Euclidean space given by

$$(x, v) \longmapsto \Big((\phi'_\beta)^{-1} \circ f \circ \phi_\alpha(x), [(d_{(\phi'_\beta)^{-1}(f(\phi_\alpha(x)))} \phi'_\beta)^{-1} \circ d_{f(x)} f \circ d_x \phi_\alpha](v)\Big)$$

Using the chain rule, we identify the right term as $d_x((\phi'_\beta)^{-1} \circ f \circ \phi_\alpha)$, which equals the total derivative $D_x((\phi'_\beta)^{-1} \circ f \circ \phi_\alpha)$. That is, we are dealing with the map

$$(x, v) \longmapsto \left((\phi'_\beta)^{-1} \circ f \circ \phi_\alpha(x), D_x((\phi'_\beta)^{-1} \circ f \circ \phi_\alpha)(v)\right).$$

This is evidently linear on each fibre and smooth. $\square$

Example 6.2.6. If $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^{k'}$ are open and $f\colon U \to V$ is a smooth map, then $df\colon TU \to TV$ is the map

$$\begin{aligned} TU = U \times \mathbb{R}^k &\longrightarrow TV = V \times \mathbb{R}^k \\ (x, v) &\longmapsto (f(x), D_x f(v)) \end{aligned}$$

obtained by applying pointwise the total derivative of f.

Using that the equations $d_m(\mathrm{id}) = \mathrm{id}$ and $d_m(g \circ f) = d_{f(m)} g \circ d_m f$ hold in each fibre, we see that:

Lemma 6.2.7. *The derivative satisfies* $d(\mathrm{id}) = \mathrm{id}$ *and* $d(g \circ f) = dg \circ df$.

6.3 Linear algebra of vector bundles

For later use, we want to generalize our usual definitions and constructions for vector spaces to vector bundles.

6.3.1 Subbundles

The generalization of a subspace of a vector space is a subbundle.

Definition 6.3.1. Let $p\colon E \to X$ be a k-dimensional vector bundle. A subspace $E' \subset E$ is a *k'-dimensional subbundle* if each $E'_x := p^{-1}(x) \cap E'$ is a k'-dimensional linear subspace of $E_x = p^{-1}(E)$ and there are local trivializations $\phi\colon \bigsqcup_{x \in U} E_x \cong U \times \mathbb{R}^k$ sending $\bigsqcup_{x \in U} E'_x$ to $U \times \mathbb{R}^{k'}$.

If (p, E, M) is a smooth vector bundle, we can make sense of a smooth subbundle, by requiring that the local trivializations are smooth.

6.3.2 Kernels

Using this we can make sense of the kernel and image of certain maps of vector bundles. This requires the following technical lemma, whose proof you do not need to know. Let $\mathrm{Lin}(\mathbb{R}^p, \mathbb{R}^{p'})$ denote the space of linear map $\mathbb{R}^{p'} \to \mathbb{R}^p$, topologised by identifying it with $\mathbb{R}^{pp'}$.

Lemma 6.3.2. *If* $\Gamma\colon \mathbb{R}^n \to \mathrm{Lin}(\mathbb{R}^p, \mathbb{R}^{p'})$ *is a continuous map whose image lies in subspace of linear maps of rank exactly equal to* r*, then there exists an open neighbourhood* $W \subset \mathbb{R}^n$ *of* 0 *and continuous maps* $B\colon W \to \mathrm{GL}_{p'}(\mathbb{R})$ *and* $C\colon W \to \mathrm{GL}_p(\mathbb{R})$ *so that* $C(w)\Gamma(w)B(w) = \Gamma(0)$ *for all* $w \in W$*. If* Γ *is smooth, then* B *and* C *can also be taken to be smooth.*

Proof. We may as well change bases to something more convenient: pick a basis of $\mathbb{R}^p$ and $\mathbb{R}^{p'}$ such that in this basis $\Gamma(0)$ is given by the $(p \times p')$-matrix (the 0's are rectangular matrices filled with 0's of the correct size)

$$\pi_r = \begin{bmatrix} \mathrm{id}_r & 0 \\ 0 & 0 \end{bmatrix}.$$

With respect to these bases, for w in an open neighbourhood W of 0 the matrix of $\Gamma(w)$ is given by

$$\pi_r + A = \begin{bmatrix} \mathrm{id}_r + A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with $||A||^2 < 1/2$ (with $||A||^2$ the sum of the squared entries). In fact, because the first r rows contains a unique entry $> 1/2$, A_{21} and A_{22} have to be 0 for this to have rank r.

We will use $C(w)$ to get rid of A_{11}:

$$C(w) = \begin{bmatrix} (\mathrm{id}_r + A_{11})^{-1} & 0 \\ 0 & \mathrm{id}_{p'-r} \end{bmatrix}$$

with the inverse in the top-right square existing because each row contains a unique entry $> 1/2$. We compute that

$$C(w)\Gamma(w) = \pi_r + A = \begin{bmatrix} \mathrm{id}_r & (\mathrm{id}_r + A_{11})^{-1}A_{12} \\ 0 & 0 \end{bmatrix}$$

We will then use $B(w)$ to get rid of the $(r \times p - r)$-matrix $(\mathrm{id}_r + A_{11})^{-1}A_{12}$: it will be the $(p \times p)$-matrix given by

$$B(w) = \begin{bmatrix} \mathrm{id}_r & -(\mathrm{id}_r + A_{11})^{-1}A_{12} \\ 0 & \mathrm{id}_{p-r} \end{bmatrix}$$

and it is a simple computation that $C(w)\Gamma(w)B(w) = \Gamma(0)$.

Since the construction of $C(w)$ and $B(w)$ depends continuously on the entries of $\Gamma(w)$ these maps are continuous. □

Lemma 6.3.3. *Suppose $p\colon E \to X$ and $p'\colon E' \to X'$ are vector bundles, and $G\colon E \to E'$ is a map of vector bundles so that $G_x\colon E_x \to E'_{g(x)}$ has the same rank for all $x \in X$. Then*

$$\ker(G) := \bigsqcup_{x \in X} \ker(G_x)$$

is a subbundle of E. If the vector bundles and the map between them are smooth, then $\ker(G)$ *is a smooth subbundle.*

Proof. Passing to local trivializations of p and p', we may assume that G is a continuous map $U \times \mathbb{R}^p \to V \times \mathbb{R}^{p'}$ over a continuous map $g\colon U \to V$ so that $G(u, -)\colon \mathbb{R}^p \to \mathbb{R}^{p'}$ is linear of fixed rank r. In other words, G is described by a g and a continuous map $\Gamma\colon U \to \mathrm{Lin}(\mathbb{R}^{p'}, \mathbb{R}^p)$ landing in the subspace of linear spaces that have rank r. By the previous lemma, on a neighbourhood of each point $u_0 \in U$ we adjust the local trivializations is that Γ is constant with value π_r. □

6.3.3 Images

The image of a vector bundle map is not defined in general. On the one hand, if the underlying map on base spaces is not injective, it will try to assign two fibres to the same point in the target. On the other hand, if the underlying map on base spaces is not surjective, it will not know what fibres to assign to some points in the target. These issues are resolved by restricting our attention to inclusions of base spaces only, and constructing the image of the vector bundle map only over the image of this inclusion.

Definition 6.3.4. Suppose that $p\colon E \to X$ is a vector bundle and $Y \subset X$ a subspace, then $E|_Y := \bigcup_{y \in Y} E_y$ with the subspace topology is a vector bundle over Y.

This definition makes sense, because the local trivializations of E restrict to local trivializations of $E|_Y$.

Example 6.3.5. The local triviality condition in the definition of a k-dimensional vector bundle $p\colon E \to X$ can rephrased as saying that for all $x \in X$ there exists an open subset $U \subset X$ such that $E|_U$ is isomorphic to the trivial bundle $U \times \mathbb{R}^k$.

A similar argument as for kernels now tells us that:

Lemma 6.3.6. *Suppose $p\colon E \to X$ and $p'\colon E' \to X'$ are vector bundles and $X \subset X'$, and $G\colon E \to E'$ over the inclusion so that $G_x\colon E_x \to E'_x$ has the same rank for all $x \in X$. Then*

$$\mathrm{im}(G) := \bigsqcup_{x \in X} \mathrm{im}(G_x)$$

is a subbundle of $E'|_X$. If the vector bundles and the map between them are smooth, then $\mathrm{im}(G)$ *is a smooth vector bundle.*

6.3.4 Quotients

Given a subspace of a vector space, we can take the quotient. Similarly, we can take the fibewise quotient of a vector bundle by a subbundle.

Lemma 6.3.7. *Let $E \to X$ be a vector bundle and $E' \subset E$ a subbundle. Then the quotients of the vector space E_x by the subspace E'_x assemble to a vector bundle*

$$E/E' := \bigsqcup_{x \in X} E_x/E'_x$$

over X using the quotient topology, which we call the quotient bundle. *If E was a smooth vector bundle and E' a smooth subbundle, then E/E' is also a smooth vector bundle.*

6.4 Problems

Problem 11 (Construction of smooth vector bundles)**.** Prove Lemma 6.1.9.

Problem 12 (Tangent bundles to submanifolds)**.** Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold.

(a) Prove that

$$T^{\mathrm{submfd}}M = \{(m, v) \in M \times \mathbb{R}^n \mid v + m \in T^{\mathrm{submfd}}_m M\}$$

is a k-dimensional smooth vector bundle.

(b) Prove that TM and $T^{\mathrm{submfd}}M$ are isomorphic as smooth vector bundles.

Chapter 7

Immersions and submersions

In this lecture we continue with implementation of one of the slogans of differential topology: *state globally, prove locally.* We do so by importing the inverse function theorem and its corollaries into the language of smooth manifolds. The main difficulty is figuring out the correct statements, as most proofs will start by passing to charts and then work on open subsets of Euclidean space.

This covers 1.§3 of [GP10], as well as a version of pages 51–52.

7.1 Globalizing the inverse function theorem

The easiest example of the above slogan is a characterisation of diffeomorphisms where you do not need to go through the effort of finding the inverse and proving it is smooth. We start by recalling the statement of the inverse function theorem [DK04a, Theorem 3.2.4]:

Theorem 7.1.1 (Inverse function theorem)**.** *Let $U_0 \subset \mathbb{R}^n$ be open and $a \in U_0$. Suppose $g\colon U_0 \to \mathbb{R}^n$ is a smooth map whose total derivative Dg_a at a is an invertible linear map. Then there exists an open neighbourhood $U \subset U_0$ of a such that $g(U)$ is open and*

$$g|_U \colon U \longrightarrow g(U)$$

is a diffeomorphism onto this open subset.

To translate this into the language of smooth manifolds we recall the notions we introduced in the previous lecture. We constructed for each k-dimensional smooth manifold M a tangent bundle TM, which is a k-dimensional smooth vector bundle over M. Each smooth map $f\colon M \to N$ with M a k-dimensional smooth manifold and N a k'-dimensional smooth manifold, induces a map of smooth vector bundles $df\colon TM \to TN$ called the derivative.

By construction, both of the tangent bundle and the derivative are easy to understand when viewed through the lens of a chart. A chart $(U_\alpha, V_\alpha, \phi_\alpha)$ of M with $p \in U_\alpha$ gives an identification the restriction of TM to V_α with $U_\alpha \times \mathbb{R}^k$. A chart $(U'_\beta, V'_\beta, \phi'_\beta)$ with $f(p) \in V'_\beta$ gives a similar identification of the restriction of TN to V'_β with $U'_\beta \times \mathbb{R}^{k'}$. Under these identifications, the derivative

$d_p f\colon T_pM \to T_{f(p)}N$ is the total derivative of $(\phi'_\beta)^{-1} \circ f \circ \phi_\alpha$ at $\phi_\alpha^{-1}(p)$. That is, the following diagram of vector spaces and linear maps commutes:

$$\begin{tikzcd} T_pM \arrow[r, "d_p f"] \arrow[d, "\cong"'] & T_{f(p)}N \arrow[d, "\cong"] \\ \mathbb{R}^p \arrow[r, "D_{\phi_\alpha^{-1}(p)}((\phi'_\beta)^{-1}\circ f\circ\phi_\alpha)"'] & \mathbb{R}^{k'}. \end{tikzcd} \tag{7.1}$$

We shall translate the hypothesis on $d_p f$ into one about the bottom linear map, and then apply the inverse function theorem to get:

Lemma 7.1.2. *Let $f\colon M \to N$ be a smooth map with M k-dimensional and N k'-dimensional, and suppose that $d_p f\colon T_pM \to T_{f(p)}N$ is an isomorphism. Then $k = k'$ and f is a local diffeomorphism at p, i.e. there is an open neighborhood V of p in M such that $f|_V\colon V \to f(V)$ is a diffeomorphism.*

Proof. Using (7.1), the hypothesis translates into the statement that the total derivative of the map

$$(\phi'_\beta)^{-1} \circ f \circ \phi_\alpha\colon U_\alpha \supset \phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V'_\beta)) \longrightarrow \phi_\alpha^{-1}(f(V_\alpha) \cap V'_\beta) \subset U'_\beta.$$

at $\phi_\alpha^{-1}(p)$ is an isomorphism. This is only possible if the total derivative is a square matrix, so $k = k'$. When we call this function g and apply the inverse function theorem to it at $a = \phi_\alpha^{-1}(p)$, we get an open subset $U \subset \phi_\alpha^{-1}(f(V_\alpha) \cap V'_\beta)$ such that $g(U)$ is open and $g|_U\colon U \to g(U)$ is a diffeomorphism. Translating this back into M and setting $V := \phi_\alpha(U)$ through the commutative diagram

$$\begin{tikzcd} V_\alpha \supset V \arrow[r, "f"] & f(V) \subset V'_\beta \\ U_\alpha \supset U \arrow[u, "\phi_\alpha"] \arrow[r, "g"] & g(U) \subset U'_\beta, \arrow[u, "\phi'_\beta"'] \end{tikzcd}$$

this is saying that $f(V) = \phi_\beta(g(U))$ is open in N and $f|_V\colon V \to f(V)$ is a diffeomorphism. ☐

Theorem 7.1.3. *A bijective smooth map $f\colon M \to N$ which has a bijective differential at all $p \in M$ is a diffeomorphism.*

Proof. Since $f\colon M \to N$ is a bijection, it has an inverse $f^{-1}\colon N \to M$. To see that this is smooth at $f(p) \in M$, apply the previous lemma and observe that on $f(V)$, f^{-1} coincides with $(f|_V)^{-1}$. The latter is smooth as the inverse of the diffeomorphism $f|_V$. ☐

Example 7.1.4. The quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a surjective smooth map which has bijective differential at all $p \in \mathbb{R}^2$, but it is not a diffeomorphism as it is not even a homeomorphism.

We can avoid having to check that f is surjective by demanding M is compact and N is connected.

Corollary 7.1.5. *If M is non-empty compact and N is connected, an injective smooth map $f\colon M \to N$ which has a bijective differential at all $p \in M$ is a diffeomorphism.*

Proof. In light of the previous theorem it remains to prove that f is surjective. By Lemma 7.1.2 the image of f is open. The image of every compact space under a continuous map is compact and in a Hausdorff space every compact set is closed, so the image of f is both open and closed. This means it is a union of connected components of N and by assumption N has a single such component, hence f must be surjective. □

7.2 Globalizing the immersion theorem

We next globalize the immersion theorem [DK04a, Section 4.3], which said:

Theorem 7.2.1 (Immersion theorem)**.** *Let $U_0 \subset \mathbb{R}^k$ be an open subset and $a \in U_0$. Suppose we have a smooth map $h\colon U_0 \to \mathbb{R}^{k'}$ such that the total derivative Dh_a of h at a is injective. Then $k \leqslant k'$ and there exist open neighbourhoods $U \subset U_0$ of a and $V \subset \mathbb{R}^n$ of $h(a)$, and diffeomorphisms $\phi\colon \mathbb{R}^k \to U$ and $\phi'\colon \mathbb{R}^{k'} \to V$ such that*

(i) $\phi(0) = a$,

(ii) $\phi'(0) = h(a)$, *and*

(iii) the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "\cong", "\phi"'] \arrow[d, "\iota_k"] & U \subset \mathbb{R}^k \arrow[d, "h"] \\ \mathbb{R}^{k'} \arrow[r, "\cong", "\phi'"'] & V \subset \mathbb{R}^{k'}, \end{tikzcd}$$

with ι_k the inclusion $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$.

Let us name the condition that the differential is injective at some point in domain:

Definition 7.2.2. Let $f\colon M \to N$ be a smooth map.

- We say f is an *immersion at p* if $d_pf\colon T_pM \to T_{f(p)}N$ is an injective linear map.
- We say f is an *immersion* if it is an immersion at all $p \in M$.

Applying the immersion theorem to $(\phi'_\beta)^{-1} \circ f \circ \phi_\alpha$ for charts $(U_\alpha, V_\alpha, \phi_\alpha)$ and $(U'_\beta, V'_\beta, \phi'_\beta)$ around p and $f(p)$ respectively, we deduce:

Lemma 7.2.3. *Let $f\colon M \to N$ be a smooth map which is an immersion at p, with M k-dimensional and N k'-dimensional. Then $k \leqslant k'$ and there exists a* *chart* $(U_\alpha, V_\alpha, \phi_\alpha)$ *of* M *around* p *and a chart* $(U'_\alpha, V'_\alpha, \phi'_\alpha)$ *of* N *around* $f(p)$ *so that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{R}^k \supset U_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, "\iota_k"] & M \arrow[d, "f"] \\
\mathbb{R}^{k'} \supset U'_\beta \arrow[r, "\phi'_\beta"] & N,
\end{tikzcd}$$

with ι_k *the inclusion onto first* k' *coordinates.*

Remark 7.2.4. Note that a linear map being injective is an open condition, which is reflected in the above lemma by the observation that if f looks like the standard inclusion in some coordinates at p, then it does so near p, namely on all of $\phi_\alpha(U_\alpha)$.

Unlike being a diffeomorphism, being an immersion is a purely local condition. This means that its image may be pathological. Of course, since an immersion need not be injective it may intersect itself, see the first example of Figure 7.1. However, even an injective immersion need not be a homeomorphism onto its image, see the second example of Figure 7.1.

Example 7.2.5. One of the worst examples is the immersion

$$\begin{aligned} h\colon \mathbb{R} &\longrightarrow \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \\ x &\longmapsto [x, \theta x] \end{aligned}$$

with $\theta \in (0,1)$ irrational. This immersion has dense image in $\mathbb{T}^2$. To see it is an immersion define $\tilde{h}(x)\colon \mathbb{R} \to \mathbb{R}^2$ by $x \mapsto (x, \theta x)$ and consider the commutative diagram of vector spaces

$$\begin{tikzcd}
T_x\mathbb{R} \arrow[r, "d_x\tilde{h}"] \arrow[dr, "d_x h"'] & T_{\tilde{h}(x)}\mathbb{R}^2 \arrow[d, "d_{\tilde{h}(x)}q"] \\
 & T_{h(x)}\mathbb{R}^2/\mathbb{Z}^2.
\end{tikzcd}$$

The linear map $d_{\tilde{h}(x)}q$ is an isomorphism because the map $\tilde{h}$ is a local diffeomorphism, and the total derivative of $\tilde{h}$ at x is easily seen to be injective, $d_x h$ must also be injective.

That is, we would like $f(M)$ not to intersect the image $\phi'_\beta(U'_\beta)$ of a chart again. If f were a homeomorphism onto its image, then $f(V_\alpha)$ would be open in $f(M)$ and this means that there is an open neighborhood V' in N such that $V' \cap f(M) = f(V_\alpha)$ so by shrinking $\phi'_\beta(U'_\beta)$ we could arrange that $\phi'_\beta(U'_\beta) \cap f(M) = f(V_\alpha)$. That such a open subset V' exists is proven by contradiction: if it did not exist then there would be a sequence of points $y_i \in f(M)\backslash f(V_\alpha)$ converging to $y \in f(M)$, which contradicts the fact that $f(V_\alpha)$ is open. In this case the charts from the immersion theorem give the image of f the structure of an r-dimensional submanifold of N. We will make this precise in a moment.

Remark 7.2.6. The advantage of the condition on an immersion being purely local is that we can classify them up to regular homotopy using an h-principle, as discussed in the first lecture.

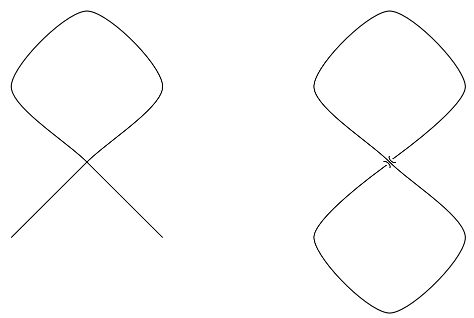


Figure 7.1 The image of two different immersions of $\mathbb{R}$ into $\mathbb{R}^2$.

7.2.1 Embeddings

Definition 7.2.7. An *embedding* is an injective immersion which is a homeomorphism onto its image.

Example 7.2.8. If m, n are integers such that $\gcd(m,n) = 1$, then the map

$$\tilde{e}\colon \mathbb{R} \ni t \longmapsto (mt, nt) \in \mathbb{R}^2$$

is easily seen to be an embedding. Taking the quotient by the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ induces an injective smooth map $e\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2/\mathbb{Z}^2$ which is automatically proper. To see this its differential is injective everywhere, we use the commutative diagram of smooth maps

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{e}"] \arrow[d] & \mathbb{R}^2 \arrow[d] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "e"] & \mathbb{R}^2/\mathbb{Z}^2 \end{tikzcd}$$

and fixing $p \in \mathbb{R}$ we get a commutative diagram of linear maps

$$\begin{tikzcd} T_p\mathbb{R} \arrow[r, "d_p e"] \arrow[d] & T_{\tilde{e}(p)}\mathbb{R}^2 \arrow[d] \\ T_{[p]}\mathbb{R}/\mathbb{Z} \arrow[r, "d_{[p]}e"] & T_{e([p])}\mathbb{R}^2/\mathbb{Z}^2. \end{tikzcd}$$

The vertical maps are isomorphisms by a previous example, and the top map is injective. Hence the bottom map is also injective.

This gives an example of many embeddings of circles into $\mathbb{T}^2$, one in each homotopy class $(m,n) \in \mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$ which $\gcd(m,n) = 1$. These are the only elements of the fundamental group which can be represented by embeddings (if we use the convention $\gcd(0,0) = 1$) [Rol90, Theorem 2.C.2].

Proposition 7.2.9. *A subset $X \subset M$ is a submanifold if and only if it is the image of an embedding.*

Proof. For $\Leftarrow$, observe that we can use the local charts provided by Lemma 7.2.3 to make $e(X)$ a submanifold. For $\Rightarrow$, it suffices to prove that the inclusion

$\iota\colon X \hookrightarrow M$ is an embedding. It is visibly a homeomorphism onto its image, and by computing locally in the charts provided by the fact ι is an immersion, we see that its differential $d\iota$ is injective everywhere. $\square$

In the proof of Proposition 7.2.9, the charts used to make $e(X)$ into a submanifold exhibit $e\colon X \to e(X)$ as a bijective smooth map which has bijective differential at all $x \in X$. By Theorem 7.1.3, e is not just a homeomorphism onto its image but a diffeomorphism. Let us record this:

Corollary 7.2.10. *If $e\colon X \hookrightarrow M$ is an embedding then it is a diffeomorphism onto its image.*

Let us discuss further the condition that an embedding is homeomorphism onto its image. If the domain of an injective immersion $X \hookrightarrow M$ is compact, it restricts to a continuous bijection $X \to \mathrm{im}(X)$ of compact Hausdorff spaces and hence is a homeomorphism onto its image. If the domain is not compact, we can instead add the following condition:

Definition 7.2.11. A continuous $f\colon X \to Y$ is *proper* if $f^{-1}(K) \subset X$ is compact for all compact $K \subset Y$.

Intuitively, a proper map is one that "maps infinity to infinity." One way to see that a map is not proper is to recall that proper maps between locally compact Hausdorff spaces are closed, allowing us to easily construct embeddings that are not proper.

Theorem 7.2.12. *A proper injective immersion is an embedding.*

Proof. It suffices to prove that if $e\colon X \to M$ is an proper injective immersion then it is a homeomorphism onto its image. Since e is presumed continuous and injective, we will use properness to deduce that e is open. Thus we need to show that if W is open in X then $e(W)$ open in $e(X)$. We will do so by contradiction, and hence suppose there is a sequence $y_1, y_2, \ldots$ in $e(X)$ but not in $e(W)$, and converging to $y \in e(W)$. As $\{y, y_1, y_2, \ldots\}$ is compact in M, so is its inverse image in X because e is proper. Thus it has an accumulation point, and by passing to a subsequence we may assume that the $e^{-1}(y_i)$ converge to some $z \in X$. Then $e(e^{-1}(y_i))$ converges both to $y \in e(W)$ and $e(z) \in e(X)$ so $y = e(z)$ and by injectivity of e thus $e^{-1}(y) = z$. But since W is open in X this means that $e^{-1}(y_i) \in W$ for i large enough, contradicting $y_i \notin e(W)$. $\square$

Corollary 7.2.13. *An injective immersion with compact domain is an embedding.*

Proposition 7.2.14. *A closed subset X is a submanifold if and only if the image of a proper embedding.*

Proof. For $\Leftarrow$, we use that proper maps are closed. For $\Rightarrow$, suppose that $K \subset M$ is compact and $\{U_i\}$ is an open cover of $\iota^{-1}(K)$. Then there exists an open cover $\{\tilde{U}_i\}$ of $X \cap K \subset M$ and since $X \cap K$ is closed inside a compact it is compact, and there is a finite subcover $\tilde{U}_1, \ldots, \tilde{U}_n$. The corresponding open subsets $U_1, \ldots, U_n$ are finite subcover of $\iota^{-1}(K)$ in X. $\square$

7.3 Globalizing the submersion theorem

We can similarly globalize the submersion theorem [DK04a, Section 4.5].

Theorem 7.3.1 (Submersion theorem). *Let $U_0 \subset \mathbb{R}^k$ be open and $a \in U_0$. Suppose we have a smooth map $g\colon U_0 \to \mathbb{R}^{k'}$ such that the total derivative Dg_a of g at a is a surjective linear map. Then $k' \leqslant k$ and there exist open neighbourhoods $U \subset U_0$ of a and $V \subset \mathbb{R}^{k'}$ of $g(a)$ and diffeomorphisms $\phi\colon \mathbb{R}^k \to U$ and $\phi'\colon \mathbb{R}^{k'} \to V$ such that*

(i) $\psi(0) = a$,

(ii) $\varphi(0) = g(a)$, and

(iii) the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "\cong", "\phi"'] \arrow[d, "\pi_{k'}"] & U \subset U_0 \subset \mathbb{R}^k \arrow[d, "g"] \\ \mathbb{R}^p \arrow[r, "\cong", "\phi'"'] & V \subset \mathbb{R}^{k'}, \end{tikzcd}$$

with $\pi_{k'}$ the projection $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{k'})$.

Definition 7.3.2. Let $f\colon M \to N$ be a smooth map.

- We say f is a *submersion at p* if $d_pf\colon T_pM \to T_{f(p)}N$ is a surjective linear map.
- We say f is a *submersion* if it is a submersion at all $p \in M$.

As before, we get:

Lemma 7.3.3. *Let $f\colon M \to N$ be a smooth map which is a submersion at p, with M k-dimensional and N k'-dimensional. Then $k' \leqslant k$ and there exists a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ of M around p and a chart $(U'_\alpha, V'_\alpha, \phi'_\alpha)$ of N around $f(p)$ so that the following diagram commutes*

$$\begin{tikzcd} \mathbb{R}^k \supset U_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, "\pi_{k'}"] & M \arrow[d, "f"] \\ \mathbb{R}^{k'} \supset U'_\beta \arrow[r, "\phi'_\beta"] & N, \end{tikzcd}$$

with $\pi_{k'}$ the projection onto first k' coordinates.

Remark 7.3.4. Note that a linear map being a submersion is open condition, which is reflected in the above lemma by the observation that if f looks like the standard projection in some coordinates at p, then it does so near p, namely on all of $\phi_\alpha(U_\alpha)$.

However, its main use is that if we denote $c := f(p)$ it says that $f^{-1}(c)$ is a $(k-k')$-dimensional submanifold near p; in the charts it is just $U_\alpha \cap \{(0, \ldots, 0, x_{k'+1}, \ldots, x_k)\}$. Furthermore, as in these chart the tangent spaces to this subset are given by the kernel of the derivative of $\pi_{k'}$, the tangent space to $f^{-1}(c)$ at p is given by the kernel of $d_p f$ when we identify it with a subspace of T_pM using the derivative of the inclusion map $f^{-1}(c) \to M$. This leads to the following definition and theorem:

Definition 7.3.5. Let $f\colon M \to N$ be a smooth map. Then a point $c \in N$ is called a *regular value of* f if f is a submersion at all $x \in f^{-1}(c)$.

Theorem 7.3.6 (Preimage theorem)**.** *If $f\colon M \to N$ is a smooth map and $c \in N$ a regular value, then $f^{-1}(c)$ is a $(k-k')$-dimensional submanifold of M and $T_pf^{-1}(c) = \ker(d_pf\colon T_pM \to T_{f(p)}M)$ for all $p \in f^{-1}(c)$.*

It is often more convenient to remember not the dimension of $f^{-1}(c)$, but how much this is smaller than the dimension of M; this is the *codimension* and in the previous theorem $f^{-1}(c)$ has codimension k'.

Example 7.3.7. If $f\colon M \to N$ is a submersion, then all points in N are regular values.

It may also be helpful to name those points in N that are *not* regular values.

Definition 7.3.8. Let $f\colon M \to N$ be a smooth map. Then a point $c \in N$ is called a *critical value of* f if it is not a regular value of f.

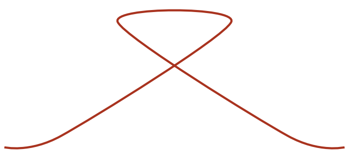
Example 7.3.9. The map $\mathbb{R}^k \to \mathbb{R}$ given by

$$(x_1, \ldots, x_k) \longmapsto x_1^2 + \ldots + x_i^2 - x_{i+1}^2 - \ldots - x_k^2$$

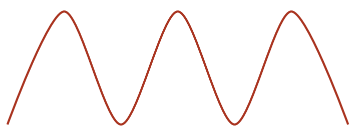
has 0 has its only critical value; all other $t \in \mathbb{R}$ are regular values.

7.4 Problems

Problem 13 (Images of immersions)**.** Are following subsets of $\mathbb{R}^2$ the image of an immersion and/or an embedding $\mathbb{R} \to \mathbb{R}^2$ (you should imagine them continuing indefinitely)? You need to explain your reasoning for each example, but do not need to give proofs.

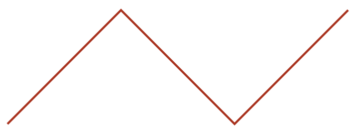


(i)



(ii)

(iii)



(iv)

Problem 14 (Submersions, immersions, and smooth maps)**.**

(a) Suppose that $f\colon M \to N$ is an immersion and $h\colon P \to M$ is a continuous map. Prove that h is smooth if and only if $f \circ h$ is.

(b) Suppose that $f\colon M \to N$ is a surjective submersion and $g\colon N \to P$ is a continuous map. Prove that g is smooth if and only if $g \circ f$ is.

Problem 15 (Submersions with compact domain)**.**

(a) Suppose $f\colon M \to N$ is a submersion with M a compact smooth manifold and N a connected smooth manifold. Show that f is surjective. (Hint: show that its image is both open and closed.)

(b) Show that there exists no submersion from a compact smooth manifold to a Euclidean space of positive dimension.

Problem 16 (A family of surfaces)**.** Prove that the subspace

$$X = \{(x, y, z) \mid (x^4 - x^2 + y^2)^2 + z^2 = \epsilon\} \subset \mathbb{R}^3$$

is a 2-dimensional smooth submanifold for $\epsilon > 0$ sufficiently small. Sketch it. What happens when we increase ϵ?

Problem 17 (Special orthogonal groups)**.** Let $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$ be the subgroup of orthogonal matrices, i.e. A such that $A^t = A^{-1}$. This is known as the *orthogonal group.*

(a) Using the submersion theorem to prove that $\mathrm{O}(n)$ is a $\frac{1}{2}n(n-1)$-dimensional manifold.

(b) Prove that $\mathrm{O}(n)$ is a Lie group.

(c) Show that $\mathrm{O}(n)$ has two path components.

The path component $\mathrm{SO}(n) \subset \mathrm{O}(n)$ containing the identity is a Lie group known as the *special orthogonal group.*

Problem 18 (Some orthogonal Stiefel manifolds)**.** Let $V_2(\mathbb{R}^n)$ be the subset of $(\mathbb{R}^n)^2$ of pairs (v_1, v_2) of vectors such that $||v_1||^2 = 1 = ||v_2||^2$ and $v_1 \cdot v_2 = 0$.

(a) Prove that $V_2(\mathbb{R}^n)$ is a smooth manifold.

(b) Prove that $V_2(\mathbb{R}^3)$ is diffeomorphic to the special orthogonal group $\mathrm{SO}(3)$.

(c) Let W_n be the subset of $\mathbb{C}^n$ of n-tuples $(z_1, \ldots, z_n)$ satisfying $z_1^2+\cdots+z_n^2 = 0$ and $|z_1|^2 + \cdots + |z_n|^2 = 2$. Prove that W_n is a smooth manifold which is diffeomorphic to $V_2(\mathbb{R}^n)$.

Problem 19 (Configuration spaces in robotics)**.** Fix an integer $n \geqslant 1$ and real numbers $r_i > 0$, $1 \leqslant r \leqslant n$. We consider the space C of configurations of a robot arm with n segments of lengths $r_1, \ldots, r_n$. We take the attaching point of the arm as the origin, and for simplicity assume that the segments are constrained to move in the plane $\mathbb{R}^2$. That is, C is the subspace of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ of points $(z_1, \ldots, z_n)$ such that $|z_i - z_{i-1}| = r_i$ for $1 \leqslant i \leqslant n$ (with the convention that $z_0 = 0$).

(a) Use the submersion theorem to show that C is a submanifold of $\mathbb{C}^n$. What is its dimension?

(b) Show that C is diffeomorphic to $(S^1)^n$.

(c) Is it still a submanifold when we add the requirement that the segments of the arm do not intersect outside the joints? That is, we take the subspace $D \subset C$ of those $(z_1, \ldots, z_n)$ such that for all $1 \leqslant i, j \leqslant n$ satisfying $i \neq j, j-1$ we have $z_i \notin \{tz_{j-1} + (1-t)z_j \mid t \in [0,1]\}$ (again with the convention that $z_0 = 0$). You have to explain your answer or give a counterexample, but do not need to give a full proof.

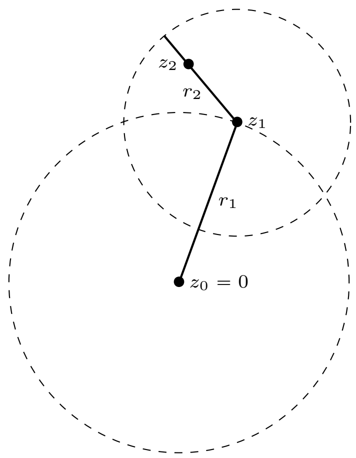


Figure 7.2 A point (z_1, z_2) in C for $n = 2$, visualized as an arm with two segments.

Problem 20 (Embeddings between projective spaces)**.** Prove that the following are smooth embeddings:

(a) The standard inclusion $\mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ induces a continuous map

$$\begin{aligned} i\colon \mathbb{R}P^n &\longrightarrow \mathbb{R}P^{n+1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_0 : \cdots : x_n : 0]. \end{aligned}$$

(b) The Segre embedding is the continuous map

$$\begin{aligned} S\colon \mathbb{C}P^1 \times \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^3 \\ ([x_0 : x_1], [y_0, y_1]) &\longmapsto ([x_0y_0 : x_1y_0 : x_0y_1 : x_1y_1]). \end{aligned}$$

Generalize this to an embedding $\mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{(i+1)(j+1)-1}$.

(c) Complexification $\mathbb{R}^n \to \mathbb{C}^n$ induces a continuous map

$$\begin{aligned} j\colon \mathbb{R}P^n &\longrightarrow \mathbb{C}P^n \\ [x_0 : \ldots : x_n] &\longmapsto [x_0 : \ldots : x_n], \end{aligned}$$

where the left hand side is an equivalence class of $(n+1)$ real numbers, which is considered as an equivalence of $(n+1)$ complex numbers on the right hand side.

Chapter 8

Quotients and coverings

In this lecture we discuss smooth manifolds which are evenly covered by another smooth manifold. Such covering maps often arise as quotients by discrete groups, and we follow with a discussion of quotients by Lie groups.

8.1 Covering spaces

In point-set topology, there is a notion of a covering of one topological space by another. One should imagine many sheets of fabric covering a surface.

Definition 8.1.1. A continuous map $p\colon E \to B$ is a *covering map* if each point $b \in B$ has an open neighbourhood U such that $p^{-1}(U)$ can be written as a union $\bigsqcup_i V_i$ of disjoint open subsets of E, such that $p|_{V_i}\colon V_i \to U$ is a homeomorphism for each i.

Example 8.1.2. Prototypical examples are

$$\begin{aligned} \mathbb{R} &\longrightarrow S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \\ t &\longmapsto e^{2\pi i t}, \end{aligned}$$

where each $z \in S^1$ has infinite pre-image, and

$$\begin{aligned} S^1 &\longrightarrow S^1 \\ z &\longmapsto z^n, \end{aligned}$$

where each $z \in S^1$ has exactly n pre-images.

Is a cover of a smooth manifold again a smooth manifold? If $p\colon E \to B$ is a covering map and B is Hausdorff or locally Euclidean then so clearly so is E. Similarly, E is second-countable when B is and p has countable fibres. Thus we know when E is a topology manifold. It remains to lift the smooth structure on B to one on E:

Theorem 8.1.3. *If $p\colon E \to B$ is a covering map such that $p^{-1}(b)$ is countable for all $b \in B$ and B is a k-dimensional smooth manifold, then there is a unique k-dimensional smooth atlas on E such that $p\colon E \to B$ is a local diffeomorphism.*

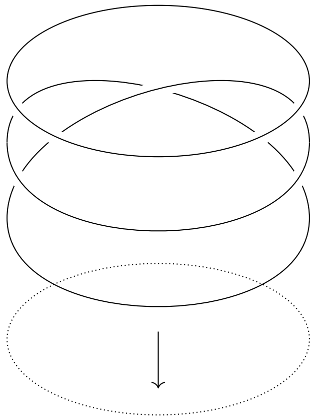


Figure 8.1 A three-fold covering of $S^1 \sqcup S^1$ by S^1.

Proof. Let us first take care of point-set topological requirements. We start by proving that E is Hausdorff when B is: $e \neq e' \in E$ with $p(e) \neq p(e')$ can be separated by $p^{-1}(U)$ and $p^{-1}(U')$ where $U, U' \subset B$ are disjoint open subsets such that $p(e) \in U$, $p(e') \in U'$. If $e \neq e' \in E$ but $p(e) = p(e')$, then they must lie in different V_i's and these open subsets separate them. To see that E is second countable, we first observe that the condition on $p^{-1}(b)$ implies that each disjoint union $\bigsqcup_i V_i$ as in Definition 8.1.1 is a countable one. Take $\{U_j\}$ a countable basis for the topology of B. By possible discarding some of the larger subsets, we may without loss of generality assume that $p^{-1}(U_j)$ is a countable union of open subsets $V_{j,i}$ of E homeomorphic to U_i. The countable collection $\{V_{j,i}\}$ is a basis for the topology of E.

We shall give a chart around each $e \in E$: pick U around $b = p(e)$ as in the definition of a covering map, and a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ around b in B such that $V_\alpha \subset U$. If V_i is such that $e \in V_i$, then we produce a chart around e by taking $U'_{\alpha,i} = U_\alpha$, taking $V'_{\alpha,i} = (p|_{V_i})^{-1}(V_\alpha)$ and setting $\phi'_{\alpha,i}$ to be

$$\phi'_{\alpha,i} \colon \mathbb{R}^k \supset U_\alpha \xrightarrow{\phi_\alpha} V_\alpha \xrightarrow{(p|_{V_i})^{-1}} V'_\beta \subset E.$$

The transition function between $(U'_{\alpha,i}, V'_{\alpha,i}, \phi'_{\alpha,i})$ and $(U'_{\beta,j}, V'_{\beta,j}, \phi'_{\beta,j})$ is only non-trivial if $V'_{\alpha,i} \cap V'_{\beta,j} \neq \varnothing$ and then it lies in V_i. Thus we can write $\phi'_{\alpha,i} = p|_{V_i}^{-1} \circ \phi_\alpha$ and $\phi'_{\beta,j} = p|_{V_i}^{-1} \circ \phi_\beta$, and the transition function is a restriction of $(p|_{V_i}^{-1} \circ \phi_\beta)^{-1} \circ (p|_{V_i}^{-1} \circ \phi_\alpha) = \phi_\beta^{-1} \circ \phi_\alpha$ and hence smooth. This completes the construction of the smooth atlas on E.

To see that p is a local diffeomorphism with respect to this smooth atlas, we use that with respect to coordinates given by the charts $(U_\alpha, V_\alpha, \phi_\alpha)$ and $(U'_{\alpha,i}, V'_{\alpha,i}, \phi'_{\alpha,i})$ it is the identity map between the equal open subsets $U'_{\alpha,i}$ and U_α of $\mathbb{R}^k$.

To see that this smooth structure is uniquely determined by this property, we must prove that the identity map of E is smooth with respect to any two smooth structures $\mathcal{A}_1, \mathcal{A}_2$ on E such that $p \colon E \to B$ is a local diffeomorphism. It suffices to verify this locally in E. The diagram

$$\begin{tikzcd}
(V_i, \mathcal{A}_1|_{V_i}) \arrow[r, "\mathrm{id}"] & (V_i, \mathcal{A}_2|_{V_i}) \arrow[d, "\cong"', "p|_{V_i}"] \\
U_i \arrow[u, "\cong", "(p|_{V_i})^{-1}"'] \arrow[r, "\mathrm{id}"] & U_i
\end{tikzcd}$$

evidently commutes, and we can think of the left map as a diffeomorphism with respect to $\mathcal{A}_1|_{V_i}$ and the right map as a diffeomorphism with respect to $\mathcal{A}_2|_{V_i}$. Since the bottom map is smooth, the top map must also be smooth. □

In fact, many local diffeomorphisms arise this way:

Proposition 8.1.4. *Suppose E and B are smooth manifolds, and $p\colon E \to B$ is a smooth map whose derivative is bijective at all points in E. If E is compact then p is a covering map.*

Proof. The conditions imply that E is a local diffeomorphism whose image is a collection of components of B so we may as well assuming p is surjective by discarding some components. For each $b \in B$, $p^{-1}(b)$ is a finite set and for each $e \in p^{-1}(b)$ the fact that p is a local diffeomorphism gives us an open subset V_e of E containing e such that $p|_{V_e}\colon V_e \to p(V_e)$ is a diffeomorphism. Using the fact that E is Hausdorff we may assume that the V_e are pairwise disjoint. Then let $U = \bigcap_e p(V_e)$, which is an open neighbourhood of b because it is a finite intersection of open subsets containing b.

We claim that $p^{-1}(U)$ is a union of the disjoint open subsets $p^{-1}(U) \cap V_e$ of E, at least after shrinking U. If so, $p|_{V_e}$ provides not just a homeomorphism $p^{-1}(U) \cap V_e \cong U$ but in fact a diffeomorphism and we would be done. We give a proof of the claim by contradiction: suppose that no matter how much we shrink U it is always the case that $p^{-1}(U)\backslash \bigcup_e V_e \neq \varnothing$. Then there exists a sequence of $x_i \in E\backslash \bigcup_e V_e$ such that the x_i converges to some $x \in E$ (since E is compact) and the $p(x_i)$ converges to b. This means that $x \in p^{-1}(b)$, and hence x_i lies in some V_e for i large enough. This gives a contradiction. □

Example 8.1.5. The Lie group $\mathrm{SO}(n)$ has a path-connected double cover $\mathrm{Spin}(n)$ for $n \geqslant 3$. Proposition 8.1.3 shows that $\mathrm{Spin}(n)$ has a unique smooth structure making $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ a local diffeomorphism.

8.2 Quotients by discrete groups

Let us discuss an important source of examples of covering maps: quotients of sufficiently nice group actions. Recall that we have an action of a discrete group G on a topological X, we always require it to be continuous in the sense that the map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned}$$

is continuous. This is equivalent to each map $g\colon X \to X$ being a homeomorphism.

Definition 8.2.1. Suppose a (discrete) group G acts on a topological space X. It acts *freely* if $gx = x$ for some $x \in X$ implies $g = e$.

We first give a condition on a free action that guarantees the quotient map $q\colon X \to X/G$ is a covering map. The following strengthening of a free action will suffice:

Definition 8.2.2. Suppose a (discrete) group G acts on a topological space X. We say it is a *covering action* if each $x \in X$ has an open neighborhood U such that $g(U) \cap U \neq \varnothing$ if and only if $g = e$.

Lemma 8.2.3. *If the action of G on X is a covering action, then the quotient map $q\colon X \to X/G$ is a covering map.*

Proof. For $q(x) \in X/G$, take the image $q(U)$ in X/G of U in Definition 8.2.2. Then $q^{-1}(q(U)) = \bigcup_g gU$ and this is a disjoint union because

$$gU \cap hU \neq \varnothing \quad \Longleftrightarrow \quad h^{-1}gU \cap U \neq \varnothing$$

and this implies $h^{-1}g = e$ so $g = h$. Furthermore, each gU is open as U is open and $g\colon X \to X$ is a homeomorphism. In particular we conclude that $q^{-1}(q(U))$ is open so $q(U)$ is open by definition of the quotient topology.

To see that the restriction of q to a map $gU \to q(U)$ is a homeomorphism, we first observe that there is a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "g", "\cong"'] \arrow[dr, "g|_U"'] & & gU \arrow[dl, "q|_{gU}"] \\ & q(U) & \end{tikzcd}$$

with horizontal map a homeomorphism. Hence it suffices to prove that this only for $g = e$. As $q|_U\colon U \to q(U)$ is clearly a continuous bijection, it remains to see it is open. But for $W \subset U$ open, $q|_U(W) \subset q(U)$ is open if and only if $q^{-1}(q|_U(W))$ is. Since $q^{-1}(q|_U(W)) = \bigcup_g gW$ this is true. □

It is clear from the definition of the quotient topology that X/G is second-countable if X is second-countable. However, it is not obvious that X/G is again Hausdorff; this requires a stronger definition:

Definition 8.2.4. Suppose a (discrete) group G acts on a topological space X. It acts *properly* if the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (x, gx) \end{aligned}$$

is proper, that is, preimages of compact subsets are compact.

Let us give some more easily verified conditions, under mild point-set topological hypotheses:

Lemma 8.2.5. *Suppose a (discrete) group G acts on a Hausdorff topological space X. This action is proper if and only if $g(K) \cap K \neq \varnothing$ for only finitely many $g \in G$ whenever $K \subset X$ is compact.*

Proof. For $\Rightarrow$, suppose there exists a compact $K \subset X$ so that there are infinitely many g_i so that $g_i(K) \cap K \neq \varnothing$. Then the action map $\alpha\colon G \times X \to X \times x$ is not proper because open cover of $\alpha^{-1}(K \times K)$ by the open subsets $\{g_i\} \times X$ admits no finite subcover.

For $\Leftarrow$, note that a compact subset $K' \subset X \times X$ is contained in the compact subset $K \times K$ for $K = \pi_1(K') \cup \pi_2(K')$. Then $\alpha(K \times K) = \bigcup_{g \in G} \{g\} \times (K \cap g(K))$ is a finite union of compact subsets so compact, and as a closed subset of this $\alpha^{-1}(K')$ is also compact. □

Lemma 8.2.6. *Suppose a (discrete) group G acts on a locally compact Hausdorff topological space X. This action is proper if and only if any two (not necessarily disjoint) $x, x' \in X$ have open neighbourhoods U, U' such that $g(U) \cap U' \neq \varnothing$ for only finitely many $g \in G$.*

Proof. For $\Rightarrow$, we apply Lemma 8.2.5 to the union $\overline{U} \cup \overline{U}'$ of disjoint compact closures of open neighbourhoods of x and x', which exist since X is locally compact Hausdorff.

For $\Leftarrow$, let $K \subset X$ be compact. We can pick for each $p = (x, x') \in K$ open neighbourhoods U of x and U' of x' so that there are only finitely many $g \in G$ so that $g(U) \cap U' \neq \varnothing$. Since K is compact, it has a cover by finitely many products of such open neighbourhoods. Suppose now that $x \in g(K) \cap K$ and write $x = g(x')$. then (x, x') lies in one of these finitely many $U \times U'$ and hence g must be among the finite collection of corresponding elements of G □

Note that smooth manifolds are always locally compact and Hausdorff.

Example 8.2.7. $\mathbb{Z}^n$ acts freely and properly on $\mathbb{R}^n$ by translation.

Example 8.2.8. If X is locally compact Hausdorff and G is *finite*, then G acts freely and properly if and only if it acts freely. To see this, observe that latter implies that for each x all elements gx for $g \in G$ are distinct. Using the Hausdorff property we can find for each $g \in G$ an open subset U_g around gx with the property that $U_g \cap U_h \neq \varnothing$ if and only if $g = h$. Then $U := \bigcap_g g^{-1}(U_g)$ is an open subset around x which satisfies $g(U) \cap U \neq \varnothing$ if and only if $g = e$. A similar argument shows that any two $x, x' \in X$ in distinct orbits have open neighbourhoods U, U' such that $g(U) \cap U' = \varnothing$ for all $g \in G$.

Proposition 8.2.9. *If G freely on a locally compact Hausdorff space X, then it acts properly if and only if $q\colon X \to X/G$ is a covering map and X/G is Hausdorff.*

Proof. For $\Rightarrow$, if the action is free in addition to being proper, for each $x \in X$ we can find an open neighbourhood V such that $g(V) \cap V \neq \varnothing$ if and only if $g = e$. To see this, take $x = x'$ in Lemma 8.2.6, $V = U \cap U'$ and shrink it using the Hausdorff property if necessary. Thus we have a covering action and by Lemma 8.2.3 says that the quotient map $q\colon X \to X/G$ is a covering map. It remains prove the quotient is Hausdorff. Take representatives x, x' of two disjoint orbits, apply Lemma 8.2.6 and use the Hausdorff property to shrink U, U' so that $g(U) \cap U' = \varnothing$ for all $g \in G$. Then $q^{-1}(q(U)) = \bigcup_{g \in G} g(U)$ and $q^{-1}(q(U')) = \bigcup_{g \in G} g(U')$ are disjoint and open, so $q(U)$ and $q(U')$ are open sets separating $[x]$ and $[x']$.

For $\Leftarrow$ we note that the right side Lemma 8.2.6 follows for x, x' in distinct orbits by finding disjoint open neighbourhoods W, W' of $[x], [x']$ using that X/G is Hausdorff, while for x, x' in the same orbit it follows from the covering property. □

If X/G happens to be a smooth manifold, this gives a smooth structure on X. We now want to go the other direction, taking X to be a smooth manifold M and assuming that the action is compatible with the smooth structure in the following sense:

Definition 8.2.10. We say that a group G *acts smoothly* on a smooth manifold M if the action map $G \times M \to M$ is smooth.

As G is discrete, this is equivalent to each $g \colon M \to M$ being a diffeomorphism. It is also equivalent to the following map being smooth

$$\begin{aligned} G \times M &\longrightarrow M \times M \\ (g, m) &\longmapsto (m, gm). \end{aligned}$$

Theorem 8.2.11. *If a discrete group G acts freely, properly, and smoothly on a k-dimensional smooth manifold M, then there is a unique k-dimensional smooth atlas on M/G such that $q \colon M \to M/G$ is a local diffeomorphism.*

Proof. We know from Lemma 8.2.9 that q is a covering map, and that M/G is Hausdorff and second countable. We next produce a smooth atlas on M/G. Let us take for each orbit $[p] \in M/G$ an open neighbourhood U as in Definition 8.1.1, so that $q^{-1}(U) = \bigsqcup_i V_i$. Let us also take charts $(U_\alpha, V_\alpha, \phi_\alpha)$ such that $V_\alpha \subset V_i$ for some i and $[p] \in q(V_\alpha)$. The charts in our atlas for M/G are then given by the $(U_\alpha, q(V_\alpha), q|_{V_i} \circ \phi_\alpha)$.

The transition function between $(U_\alpha, q(V_\alpha), q|_{V_i} \circ \phi_\alpha)$ an $(U_\beta, q(V_\beta), q_{V_j} \circ \phi_\beta)$ has non-empty domain and target if and only if $q(V_\alpha) \cap q(V_\beta) \neq \varnothing$, which happens only if $V_\alpha \cap q^{-1}(q(V_\alpha) \cap q(V_\beta)) \subset V_i$ and $V_\beta \cap q^{-1}(q(V_\alpha) \cap q(V_\beta)) \subset g(V_i)$ for some $g \in G$. Hence for the sake of computing transition functions we may replace $q|_{V_j}$ by $q|_{g(V_i)}$. Then the transition function is given by

$$(q|_{g(V_i)} \circ \phi_\beta)^{-1} \circ (q|_{V_i} \circ \phi_\alpha) = \phi_\beta^{-1} \circ g \circ \phi_\alpha,$$

which is smooth by the assumption that $g \colon M \to M$ is a diffeomorphism. This completes the construction of the smooth structure on M/G.

To see q is a local diffeomorphism with respect to this smooth structure, we use that in the local coordinates given by the charts $(U_\alpha, V_\alpha, \phi_\alpha)$ and $(U_\alpha, q(V_\alpha), q \circ \phi_\alpha)$ it is the identity map of $U_\alpha \subset \mathbb{R}^k$.

To see that this smooth structure is uniquely determined by this property, we must prove that the identity map of M/G is smooth with respect to any two smooth structures $\mathcal{A}_1, \mathcal{A}_2$ on M/G such that $q\colon M \to M/G$ is a local diffeomorphism. The diagram

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}"] \arrow[d, "q"] & M \arrow[d, "q"] \\ (M/G, \mathcal{A}_1) \arrow[r, "\mathrm{id}"] & (M/G, \mathcal{A}_2) \end{tikzcd}$$

evidently commutes, and we can think of the left map as a local diffeomorphism with respect to $\mathcal{A}_1$, of the right map as a local diffeomorphism with respect to $\mathcal{A}_2$. Since the top map is smooth, the top-right composite is. Since q is a submersion, the bottom map must also be smooth. □

Example 8.2.12. Since $\mathbb{Z}^n$ acts freely, properly, and smoothly on $\mathbb{R}^n$ by translation, Theorem 8.2.11 gives another way to construct the smooth structure on the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Example 8.2.13. Fix two coprime integers p and q. Let $\mathbb{Z}/p$ act on $S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$ by

$$k \cdot (z_1, z_2) = (e^{2\pi i k/p} z_1, e^{2\pi i q k/p} z_2).$$

This is a free smooth action of the finite group $\mathbb{Z}/p$ on the 3-dimensional smooth manifold S^3, so by Theorem 8.2.11, $L(p, q) := S^3/(\mathbb{Z}/p)$ is again a 3-dimensional smooth manifold. These are *lens spaces.* As an example, let us take $L(2, 1)$. This is the quotient of S^3 by the equivalence relation generated by $(z_1, z_2) \sim (-z_1, -z_2)$, so is diffeomorphic to $\mathbb{R}P^3$.

Example 8.2.14. Define the *configuration space of n ordered particles* in a manifold M as

$$\mathrm{Conf}_n(M) := \{(m_1, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ if } i \neq j\}.$$

As an open subset of a finite product of manifolds, this has a canonical smooth structure. The permutation action on M^n by the symmetric group $\mathfrak{S}_n$ is proper and smooth, but not free. The subset $\mathrm{Conf}_n(M)$ exactly consists of all free orbits, so the restriction of this action to $\mathrm{Conf}_n(M)$ is smooth, proper, and free. Thus the *configuration space of n unordered particles*

$$C_n(M) := \mathrm{Conf}_n(M)/\mathfrak{S}_n$$

again has a canonical smooth structure.

8.3 Quotients by Lie groups

Above we gave conditions on an action of a discrete group G on a smooth manifold M, so that the quotient M/G is again a smooth manifold. What can we say if we instead we take G to be a Lie group? The definitions, when phrased correctly, go through without modification: as before, we say that G *acts smoothly* on M if the map

$$\begin{aligned} G \times M &\longrightarrow M \times M \\ (g, m) &\longmapsto (m, gm) \end{aligned}$$

is smooth, it *acts properly* if this map is proper, and *acts freely* if the action is free. A generalization of Theorem 8.2.11 to Lie groups is the following, which we shall not prove [Lee13, Theorem 21.10]:

Theorem 8.3.1. *If a Lie group G of dimension r acts freely, properly, and smoothly on a k-dimensional smooth manifold M, then there is a unique $(k-r)$-dimensional smooth atlas on M/G such that $q\colon M \to M/G$ is a submersion.*

Example 8.3.2 (Complex projective space as quotients). The Lie group $\mathbb{C}^\times$ of non-zero complex numbers under multiplication acts freely, properly, and smoothly on $\mathbb{C}^n \backslash \{0\}$. Its quotient

$$\mathbb{C}P^n = (\mathbb{C}^n \backslash \{0\})/\mathbb{C}^\times$$

is thus a smooth manifold, giving a construction of the complex projective plane without having to give charts by hand. We will leave it an exercise for the reader to verify this construction is diffeomorphic to the previous one.

In many application we fix a Lie group G, as well as a Lie subgroup $H \subset G$, which is a subgroup which is also a smooth submanifold. It is evident that the action of H on G by multiplication is smooth and free. Furthermore, as H must be closed [Lee13, Corollary 15.30] it follows that the action is proper. The above theorem says that G/H is a smooth manifold and the quotient map

$$G \longrightarrow G/H$$

is a submersion.

Example 8.3.3 (Orthogonal Stiefel manifolds). Recall that the orthogonal Stiefel manifold $V_2(\mathbb{R}^n)$ is the submanifold of $\mathbb{R}^{2n}$ given by pairs (u, v) of orthogonal vectors in $\mathbb{R}^n$ of length 1. If we identify $O(n-2)$ be the subgroup of $O(n)$ as

$$O(n-2) \ni A \longmapsto \begin{bmatrix} A & 0 \\ 0 & \mathrm{id}_2 \end{bmatrix}$$

which is also the subgroup which fixes the vectors e_{n-1}, e_n. This identifies it as the stabiliser of this point of the transitive action of $\mathrm{O}(n0$ on $V_2(\mathbb{R}^n)$, so we get an identification

$$V_2(\mathbb{R}^n) \cong \mathrm{O}(n)/\mathrm{O}(n-2).$$

This gives another construction of the left side as a smooth manifold, which is diffeomorphic to its description as a submanifold. Replacing $n-2$ by $n-r$, we obtain more generally the *orthogonal Stiefel manifold*

$$V_r(\mathbb{R}^n) := \mathrm{O}(n)/\mathrm{O}(n-r)$$

of orthogonal frames of r vectors in $\mathbb{R}^n$. Replacing orthogonal groups by general linear groups we similarly obtain ordinary *Stiefel manifolds*.

8.4 Problems

Problem 21 (Higher-dimensional lens spaces)**.** Fix an integer p and integers $q_1, \dots, q_n$ coprime to p. The *higher-dimensional lens space* $L(p, q_1, \dots, q_n)$ is the quotient of $S^{2n-1} = \{(z_1, \dots, z_n) \mid |z_1|^2 + \cdots + |z_n|^2 = 1\} \subset \mathbb{C}^n$ by the smooth action

$$k \cdot (z_1, \dots, z_n) = (e^{2\pi i q_1 k/p} z_1, \dots, e^{2\pi i q_n k/p} z_n).$$

Prove this admits a unique smooth structure such that the quotient map $q\colon S^{2n-1} \to L(p, q_1, \dots, q_n)$ is a local diffeomorphism.

Problem 22 (Dold manifolds)**.** Let $\mathbb{Z}/2$ act on $S^m \times \mathbb{C}P^n$ by multiplication by -1 on S^m and by complex conjugation on $\mathbb{C}P^n$. Prove that

$$D(m, n) := (S^m \times \mathbb{C}P^n)/\mathbb{Z}/2$$

is a smooth manifold. This is called a *Dold manifold*.

Problem 23 (Orthogonal Grassmannians)**.** We can $\mathrm{O}(r) \times \mathrm{O}(n-r)$ with a subgroup of $\mathrm{O}(n)$ by

$$\mathrm{O}(r) \times \mathrm{O}(n-r) \ni (A, B) \longmapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathrm{O}(n).$$

(a) Show that the quotient

$$\mathrm{Gr}_r(\mathbb{R}^n) := \mathrm{O}(n)/(\mathrm{O}(r) \times \mathrm{O}(n-r))$$

is a smooth manifold.

(b) Use Gram–Schmidt to explain why we can think of $\mathrm{Gr}_r(\mathbb{R}^n)$ as a smooth manifold of r-dimensional linear subspaces of $\mathbb{R}^n$.

The smooth manifold $\mathrm{Gr}_r(\mathbb{R}^n)$ is called the *orthogonal Grassmannian of r-planes in $\mathbb{R}^n$*.

Chapter 9

Three further examples of manifolds

In these additional notes we describe three more manifolds, each interesting and an example of a more general construction.

9.1 The Poincaré homology sphere

We start with one of the first manifolds ever described, due to Poincaré. For more constructions, see [KS79].

9.1.1 The quaternions

Our construction starts with the quaternions $\mathbb{H}$. These are an associative $\mathbb{R}$-algebra, generated as an $\mathbb{R}$-vector space by elements $1, i, j, k$ which satisfy the relations

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji,\ ik = -ki,\ jk = -kj$$

$$ij = k,\ jk = i,\ ki = j.$$

This is visibly not commutative, e.g. $ij = k$ but $ji = -k$. The elements which commute with every other element, the *center*, is given by $\mathbb{R} \cdot 1$. As a $\mathbb{R}$-vector space, it is 4-dimensional, with a basis given by $1, i, j, k$.

This is a so-called division algebra, which means that algebraically it behaves like a non-commutative four-dimensional version of the complex numbers. Firstly, the quaternions have a *conjugation* operation

$$\overline{a + bi + cj + dk} := a - bi - dj - ck.$$

Lemma 9.1.1. *Conjugation is linear and an antihomomorphism, i.e. satisfies* $\overline{xy} = \overline{y}\overline{x}$.

In terms of this, we define $||x||^2 := x\overline{x}$. Explicitly, this is given by

$$||a + bi + cj + dk|| := \sqrt{a^2 + b^2 + c^2 + d^2},$$

and hence is visibly a norm (in fact the usual Euclidean one).

Every non-zero element of $\mathbb{H}$ has a unique multiplicative inverse, which can be written in terms of the conjugation and norm

$$x^{-1} = \frac{\bar{x}}{||x||^2}.$$

The 3-sphere as a Lie group

The subset $S^3 \subset \mathbb{H}$ of quaternions with norm 1 is a smooth manifold; it is just the subspace $\{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\} \subset \mathbb{H}$. The multiplication and inversion of $\mathbb{H}$ restrict to S^3. This uses the following lemma:

Lemma 9.1.2. $||xy|| = ||x||||y||$

Proof. Since the conjugation is an anti-homomorphism, we have

$$||xy||^2 = xy\overline{xy} = xy\bar{y}\bar{x} = ||x||^2||y||^2. \qquad \square$$

This exactly says that the product of two elements of norm 1 has norm 1. It also implies that the inverse of an element of norm 1 has norm 1: more generally, if $x \neq 0$ we have

$$1 = ||1|| = ||xx^{-1}|| = ||x||||x^{-1}||,$$

so $||x^{-1}|| = ||x||^{-1}$.

To see that both multiplication and inverse are smooth maps on $\mathbb{H}\backslash\{0\}$, observe they are given by polynomials in a, b, c, d. In fact, inverse is particularly easy: $g^{-1} = \bar{g}$. Hence their restriction to the submanifold S^3 is also smooth, and we conclude that S^3 is a Lie group.

Remark 9.1.3. S^1 and S^3 are the only spheres that admit the structure of a Lie group.

Example 9.1.4. In fact, this is isomorphic to the Lie group $SU(2)$ of unitary (2×2)-matrices with complex entries and determinant 1. The correspondence is given by thinking of a quaternion $a + bi + cj + dk \in \mathbb{H}$, on which S^3 acts, as a pair $(a + bi, c + di)$ of complex numbers, on which $SU(2)$ acts. Explicitly, the isomorphism of Lie groups is given by

$$S^3 \ni a + bi + cj + dk \longmapsto \begin{bmatrix} a + bi & c + di \\ -c + di & -a - bi \end{bmatrix} \in SU(2).$$

9.1.2 The Poincaré homology sphere via the binary icosahedral group

It follows from our results about quotients of manifolds by discrete groups that if $G \subset S^3$ is a finite subgroup, S^3/G admits a 3-dimensional smooth structure such that the quotient map

$$S^3 \longrightarrow S^3/G$$

is a local diffeomorphism.

Example 9.1.5. Taking $G = \{\pm 1\}$, we obtain $S^3/\{\pm 1\} = \mathbb{R}P^3$.

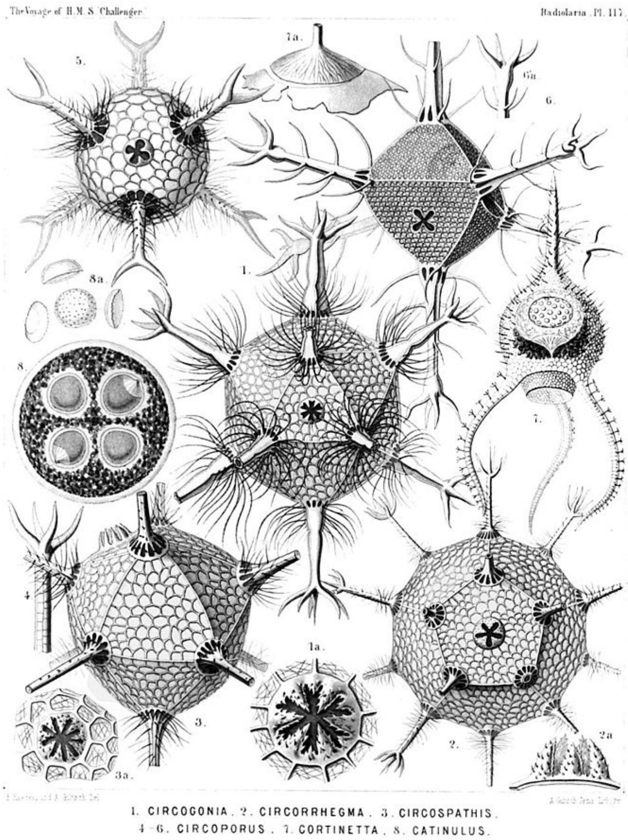


Figure 9.1 Haeckel's *"Fig. 1: Circogonia icosahedra, n. sp., × 80. The entire shell, with twelve radial tubes and twenty triangular faces. In the centre of one face is the mouth, with six teeth."* (from https://en.wikisource.org/wiki/Report_on_the_Radiolaria/Plates12#/media/File:Radiolaria_(Challenger)_Plate_117.jpg.

Our next goal is construct a particular rather large finite subgroup of S^3. The first observation is that for $g \in S^3$ the conjugation

$$S^3 \ni h \longmapsto ghg^{-1} \in S^3$$

preserves the subset of quaternions of the form $bi + cj + dk$.

We can identify this subset with $\mathbb{R}^3$ through $bi + cj + dk \longleftrightarrow (b, c, d)$. Under this identification the norm on $\mathbb{H}$ corresponds to the Euclidean norm, and thus we get an action of S^3 on $\mathbb{R}^3$ which is orthogonal. The resulting homomorphism $S^3 \to SO(3)$ has kernel of order 2. That the kernel has order at least 2 is easy to see: both $x, -x \in \mathbb{H}$ map to the same linear transformation. We leave it as an exercise to the reader that there are no further elements in the kernel.

Let the icosahedral group $I \subset SO(3)$ be the subgroup of symmetries of the icosahedron, and let I^* be its inverse image in S^3. I^* has order 120. The quotient manifold is the *Poincaré homology sphere*:

$$P := S^3/I^*.$$

Remark 9.1.6. Why is the Poincaré homology sphere interesting? As you might expect, it was first constructed by Poincaré, though he did not construct it this way. Poincaré produced it as a counterexample to the first version of the Poincaré conjecture: it has the same homology as a 3-sphere, but it is not homeomorphic to S^3 because it has fundamental group isomorphic to I^*. The correct Poincaré conjecture says that a 3-dimensional differentiable manifold that is homotopy equivalent to S^3 is diffeomorphic to it. This was eventually proven by Perelman in a series of papers in 2002–2003, for which he received a Fields medal.[1]

Remark 9.1.7. For a while some scientists thought the cosmic microwave background radiation was most consistent with the universe having space-like direction S^3/I^* instead of $\mathbb{R}^3$, though with the acquisition of more data this is no longer the case.[2]

9.2 The K3-manifold

Our second example come from algebraic geometry, and is a particular case of a general construction of a hypersurface in complex projective space.

Recall the complex projective spaces $\mathbb{C}P^k$, defined as

$$\mathbb{C}P^k = (\mathbb{C}^{k+1}\backslash\{0\})/\sim,$$

where the equivalence relation $\sim$ is generated by $(z_0, \ldots, z_k) \sim (\lambda z_0, \ldots, \lambda z_k)$ for $\lambda \in \mathbb{C}\backslash\{0\}$. In other words, we are taking the quotient of the free action of the non-zero invertible complex numbers $\mathbb{C}^\times$ by scalar multiplication on $\mathbb{C}^{k+1}\backslash\{0\}$. We denote the equivalence class of $(z_0, \ldots, z_k)$ by $[z_0 : \cdots : z_k]$. It is a $2k$-dimensional smooth manifold, covered by the $k+1$ charts

$$\begin{aligned} \phi_j \colon \mathbb{C}^k &\longrightarrow \mathbb{C}P^k \\ (z_1, \ldots, z_k) &\longmapsto [z_1 : \cdots : z_{j-1} : 1 : z_j : \cdots : z_k]. \end{aligned}$$

The image V_j of ϕ_j is given by $\{[z_0 : \ldots : z_k] \mid z_j \neq 0\}$.

Suppose we are interested in subsets of $\mathbb{C}P^n$ given by points which satisfy some equation, e.g. $f(z_0, \ldots, z_k) = 0$. Whether or not a point $[z_0 : \ldots : z_k]$ satisfies this equation ought to be independent of the choice of representative, and one way to guarantee this is the case is to assume that f is *homogeneous*:

$$f(\lambda z_0, \ldots, \lambda z_k) = \lambda^d f(z_0, \ldots, z_k)$$

for some $d \geqslant 1$. If so, if f vanishes on all representatives of $[z_0, \ldots, z_k]$ when it vanishes on one of them.

We shall now restrict our attention to such f which are polynomial, *homogeneous polynomials*. These are polynomials in $z_0, \ldots, z_k$ in which every term has the same total degree d.

[1] See https://www.ams.org/notices/200310/fea-milnor.pdf for the history and context of this problem.

[2] See e.g. http://www.ams.org/notices/200406/fea-weeks.pdf and https://mathoverflow.net/a/9717/798.

Example 9.2.1. The polynomial $z_0^2 + z_1^2$ of z_0, z_1 is homogeneous, but $z_0 + z_1^2$ is not.

We now use the submersion theorem to answer the following question: when does the zero set of homogeneous polynomial describe a smooth submanifold of $\mathbb{C}P^k$?

Theorem 9.2.2 (Hypersurfaces in complex projective spaces)**.** *Let p be a homogeneous polynomial of $z_0, \dots, z_k$ such that*

$$\{(z_0,\dots,z_k) \,|\, p(z_0,\dots,z_k) = 0\} \cap \bigcap_{j=0}^{k} \left\{(z_0,\dots,z_k) \,\middle|\, \tfrac{\partial}{\partial z_j} p(z_0,\dots,z_k) = 0\right\} = \{0\},$$

then the subspace

$$\{[z_0 : \dots : z_k] \mid p(z_0,\dots,z_k) = 0\} \subset \mathbb{C}P^k$$

is a $(2k-2)$-dimensional smooth submanifold.

This statement requires an explanation. We can identify the domain $\mathbb{C}^{k+1}$ with $\mathbb{R}^{2k}$ by $z_j \longleftrightarrow x_j + iy_j$, and similarly identify the target $\mathbb{C}$ with $\mathbb{R}^2$. Then p is not only differentiable as a function $\mathbb{R}^{2k+2} \to \mathbb{R}^2$, is in fact complex-differentiable as a function $\mathbb{C}^{k+1} \to \mathbb{C}$. That is, for each $1 \leqslant i \leqslant k$ the limit $\frac{p(z_0,\dots,z_j+h,\dots,z_k)}{h}$ with as $\mathbb{C} \ni h \to 0$ exists, and these limits are the partial derivatives $\frac{\partial p}{\partial z_j}(z_0,\dots,z_k)$.

Proof. Let us write $X := \{[z_0 : \dots : z_k] \mid p(z_0,\dots,z_k) = 0\}$. If suffices to prove that $X \cap V_j$ is a smooth submanifold for all $0 \leqslant j \leqslant k$. To do so, we may pass to the local coordinates provided by the chart ϕ_j, i.e. prove that $\phi_j^{-1}(X \cap V_j) \subset \mathbb{C}^k$ is a smooth submanifold. This is given by the vanishing set of the polynomial q_j given by $p(z_1,\dots,z_{j-1},1,z_j,\dots,z_k)$ of the k variables $z_1,\dots,z_k$ (it is not homogeneous).

We now ought to identify the domain $\mathbb{C}^k$ with $\mathbb{R}^{2k}$ and the target $\mathbb{C}$ with $\mathbb{R}^2$, and show that when $q_j(x_1+iy_1,\dots,x_k+iy_k) = 0$, the $(2 \times 2k)$-matrix of partial derivatives of the real and imaginary part of q_j with respect to $x_1,\dots,x_k$ and $y_1,\dots,y_k$ is surjective. However, it is more convenient not to leave the world of complex numbers, as q_j is complex-differentiable with respect to the k complex variables $z_1,\dots,z_k$. In this case, we can form a $(1 \times k)$-matrix of complex numbers

$$\begin{bmatrix} \frac{\partial q_j}{\partial z_1}(z_1,\dots,z_k) & \cdots & \frac{\partial q_j}{\partial z_k}(z_1,\dots,z_k). \end{bmatrix}$$

This is surjective if and only if the $(2 \times 2k)$-matrix with real entries mentioned before is surjective.

Thus the condition is that when q_j vanishes, at least one of the partial derivatives of q_j does not vanish. We will get a contradiction with the hypothesis from the assumption that q_j and all its partial derivatives vanish simultaneously. We start by relating these vanishing for q_j and partial derivatives back to p:

$$\begin{aligned} q_j \text{ vanishes at } (z_1,\dots,z_k) &\iff p \text{ vanishes at } (z_1,\dots,z_{j-1},1,z_j,\dots,z_k), \\ \tfrac{\partial q_j}{\partial z_r} \text{ vanishes at } (z_1,\dots,z_k) &\iff \tfrac{\partial p}{\partial z_{r'}} \text{ vanishes at } (z_1,\dots,z_{j-1},1,z_j,\dots,z_k), \end{aligned}$$

with $r' = r$ if $r < j$ and $r' = r+1$ if $r \geqslant j$. This gives us information about all partial derivatives except $\frac{\partial p}{\partial z_j}$.

To understand this remaining partial derivative, we use a fact due to Euler:

$$\sum_{j=0}^{k} \frac{\partial p}{\partial z_j}(z_0, \ldots, z_k) \cdot z_j = d \cdot p(z_0, \ldots, z_k), \tag{9.1}$$

with d the degree of p. To prove this, consider the function $p(\lambda z_0, \ldots, \lambda z_k) - \lambda^d p(z_0, \ldots, z_k)$. This vanishes identically because p is homogeneous of degree d, hence so its derivative with respect to λ. Evaluating this derivative at $\lambda = 1$ gives (9.1). If we use this at the point $(z_1, \ldots, z_{j-1}, 1, z_j, \ldots, z_k)$, we know the right hand side vanishes as do all terms on the left hand side expect one. We get that

$$\frac{\partial p}{\partial z_j}(z_1, \ldots, z_{j-1}, 1, z_j, \ldots, z_k) = 0,$$

which contradicts the hypothesis. This completes the proof. □

Remark 9.2.3. Implicitly we used the complex version of the submersion theorem, [DK04a, Section 3.7].

A smooth manifold obtained as in Theorem 9.2.2 is called a *hypersurface*. The example which plays such an important role in algebraic geometry is the *K3-manifold*,[3] also known as the *Fermat quartic*. It is obtained by taking the homogeneous polynomial p given by $z_0^4 + z_1^4 + z_2^4 + z_3^4$:

$$K3 \coloneqq \{[z_0 : \cdots : z_3]) \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{C}P^3.$$

It is easy to verify that the polynomial p satisfies the conditions in Theorem 9.2.2, so this is a 4-dimensional smooth manifold: if $\frac{\partial}{\partial z_j} p(z_0, z_1, z_2, z_3) = 0$ then $z_j = 0$, so all partial derivatives vanish simultaneously only at the origin.

Remark 9.2.4. Why is the K3 manifold interesting? It plays an important role in algebraic geometry and the study of 4-dimensional smooth manifolds.

When one does algebraic geometry over $\mathbb{C}$, out of a smooth k-dimensional variety one can extract a smooth $2k$-dimensional manifold ("taking the analytic topology"). In particular, the K3 manifold can be obtained this way from not one but many algebraic surfaces. There are roughly three types of algebraic surfaces: *Fano surfaces* (which are "easy"), *surfaces of general type* (which are "hard"), and *Calabi–Yau surfaces* (which are "intermediate"). The latter class contains only complex 2-dimensional tori and the K3 surfaces, and all K3 surfaces have the same underlying 4-dimensional smooth manifold: the K3 manifold that we constructed above.

Because it has an algebraic origin, the gauge-theoretic invariants used to study exotic smooth structures on smooth 4-manifolds can be computed for $K3$ using more algebraic approaches. This gives a starting point for constructing exotic smooth 4-manifolds: start with $K3$, make a modification to it, and study how this changes the gauge-theoretic invariants.

[3] The name is due to Andre Weil, who motivated it by "In the second part of my report, we deal with the Kähler varieties known as K3, named in honor of Kummer, Kähler, Kodaira and of the beautiful mountain K2 in Kashmir."

9.3 The Whitehead manifold

Our final example is quite peculiar. It is an example of a 3-dimensional smooth manifold which from the perspective of algebraic topology looks like $\mathbb{R}^3$, but is not in fact diffeomorphic to it. It is an example of infinite phenomena leading to pathological objects in differential topology.

We start with the following injective immersion $S^1 \times \mathbb{R}^2 \to \mathbb{R}^3$ of an open torus. Let us denote its complement in $\mathbb{R}^3$ by W_1. This contains another, curiously linked, open torus; its complement in $\mathbb{R}^3$ is denoted by W_2. We can keep iterating this procedure, finding a linked copy of $S^1 \times \mathbb{R}^2$ in the previous copy of $S^1 \times \mathbb{R}^2$, and denoting its complement by W_n.

The *Whitehead manifold* is then defined to be increasing union

$$W := \bigcup_n W_n.$$

This is an open subset of $\mathbb{R}^3$ and hence a smooth 3-dimensional manifold. It is the complement of the intersection of all the linked open tori, which is known as the *Whitehead continuum*.

Remark 9.3.1. Why is the Whitehead manifold interesting? The Whitehead manifold is a contractible 3-dimensional smooth manifold which is not diffeomorphic or even homeomorphic to $\mathbb{R}^3$. (Surprisingly, it is homeomorpic to a union of two copies of $\mathbb{R}^3$ intersecting in another copy of $\mathbb{R}^3$ [Gab11].)

The reason is that being contractible does not take into account the "topology at infinity," i.e. how $W \backslash K_n$ behaves as for a sequence K_n of compact codimension 0 submanifolds exhausting W. This is a general phenomenon: if you want to use algebraic topology to study non-compact manifolds you need to take into account the topology at infinity.

9.4 Problems

Problem 24 (Klein quartic)**.** Prove that the subspace

$$X = \{[x : y : z] \in \mathbb{C}P^2 \mid x^3y + y^3z + z^3x = 0\} \subset \mathbb{C}P^2$$

is a 2-dimensional compact submanifold. It is called the *Klein quartic*. What is its genus?

Problem 25 (Milnor manifolds)**.** Let $m \leqslant n$. Prove that the subspaces

$$H(m, n) := \left\{ ([z_0, \ldots, z_m], [w_0, \ldots, w_n]) \,\middle|\, \sum_{j=0}^{m} z_j w_j = 0 \right\} \subset \mathbb{C}P^m \times \mathbb{C}P^n$$

are $2(m + n - 1)$-dimensional smooth submanifolds. These are called *Milnor manifolds*.

Chapter 10

Partitions of unity and the weak Whitney embedding theorem

In this lecture we prove that every compact manifold can be embedded into a Euclidean space, using partitions of unity.

10.1 The weak Whitney embedding theorem

We now prove that every compact smooth manifold M arises a smooth submanifold of some $\mathbb{R}^N$, by constructing a smooth embedding $M \hookrightarrow \mathbb{R}^N$. The result is true also for non-compact smooth manifolds, but proving that requires more care and will be done later. Thus we could have set up the theory by demanding every smooth manifold is of this form, as [GP10] does.

The new tool in our argument is the existence of partitions of unity, and this is one of the reasons that we demanded M was second-countable and Hausdorff. Recall that the *support* $\mathrm{supp}(\eta) \subseteq M$ of a continuous function $\eta\colon M \to [0,1]$ is the closure of the open subset $\eta^{-1}((0,1])$.

Definition 10.1.1. Let $\mathcal{W} = \{W_i\}_{i \in I}$ be an open cover of M. Then a *partition of unity subordinate to* $\mathcal{W}$ is a collection of smooth function $\eta_i\colon M \to [0,1]$ with the following properties:

(i) $\mathrm{supp}(\eta_i) \subseteq W_i$,

(ii) each $p \in M$ has an open neighbourhood on which only finitely many η_i are non-zero,

(iii) for all $p \in M$, $\sum_i \eta_i(p) = 1$.

Theorem 10.1.2. *Every open cover $\mathcal{W} = \{W_i\}_{i \in I}$ of M admits a subordinate partition of unity.*

The main use of partitions of unity is to construct a function (or something similar) on W_i, usually the codomain of a chart, multiply it with η_i and extend the result by 0 elsewhere. The result is then defined on all of M.

Theorem 10.1.3 (Whitney)**.** *Every compact k-dimensional smooth manifold M has an embedding into some Euclidean space $\mathbb{R}^N$.*

Proof. Since M is compact it can be covered by the codomains V_i of finitely many charts (U_i, V_i, ϕ_i) for $1 \leqslant i \leqslant r$. Let $\eta_i \colon M \to \mathbb{R}$ be a subordinate partition of unity subordinate to this cover. We then define

$$\overline{\eta_r(p)\phi_r^{-1}}(p) := \begin{cases} \eta_r(p)\phi_r^{-1}(p) & \text{if } p \in V_i, \\ 0 & \text{otherwise.} \end{cases}$$

(Thus $\overline{\eta_r(p)\phi_r^{-1}}$ ought to be interpreted as a compound symbol.) This is smooth as the support of η_i is contained in V_i and η_i is smooth.

Then we define the following map

$$\begin{aligned} \rho\colon M &\longrightarrow \mathbb{R}^{r(k+1)} \\ p &\longmapsto (\eta_1(p), \overline{\eta_1(p)\phi_1^{-1}}(p), \ldots, \eta_r(p), \overline{\eta_r(p)\phi_r^{-1}}(p)). \end{aligned}$$

Since each of the components of ρ is smooth, so is ρ.

We must now verify ρ is injective and has injective differential for all $p \in M$ (it is automatically proper because M is compact). We start with injectivity and suppose that $\rho(p) = \rho(p')$. Since the η_i are a partition of unity we can pick an η_i such that $\eta_i(p) = \eta_i(p') \neq 0$. From this we deduce that both p and p' are in V_i. We can then divide the equation $\eta_i(p)\phi_i^{-1}(p) = \eta_i(p')\phi_i^{-1}(p')$ by $\eta_i(p) \neq 0$ to get $\phi_i^{-1}(p) = \phi_i^{-1}(p')$ and apply the injective map ϕ_i to deduce $p = p'$.

Next we verify ρ has injective differential everywhere. Let $p \in M$ be such that $\eta_i(p) \neq 0$ and set $q = \phi_i^{-1}(p)$. Since projections are smooth and on $\eta_i^{-1}((0,1])$ division by η_i is a smooth map, the following is a smooth map $\eta_i^{-1}((0,1]) \to \mathbb{R}^k$:

$$q \longmapsto \rho(q) \overset{\text{proj}}{\longmapsto} \eta_i(q)\phi_i^{-1}(q) \overset{\text{divide}}{\longmapsto} \phi_i^{-1}(q).$$

It is visibly equal to ϕ_i, so it has bijective differential $d_p\phi_i$ at p. By the chain rule we can write

$$d_p\phi_i = d_{\rho(p)}(\text{divide} \circ \text{proj}) \circ d_p\rho$$

and since the left hand side is bijective the term $d_p\rho$ on the right hand side must be injective. □

Example 10.1.4. The embeddings produced by Theorem 10.1.3 have a target of unnecessarily high dimension. For example, at best it produces an embedding of S^n into $\mathbb{R}^{2n+2}$, even though we know S^n can be embedded into $\mathbb{R}^{n+1}$. We shall later prove that every compact k-dimensional manifold embeds into $\mathbb{R}^{2k+1}$.

10.1.1 Tangent bundles of submanifolds

Suppose M is a k-dimensional manifold and $Z \subset M$ is a submanifold of codimension r. Then both M and Z have tangent bundles TM and TZ. The inclusion $i\colon Z \hookrightarrow M$ is an injective map whose derivative is injective at all $z \in Z$. Thus the map $di\colon TZ \to TM$ is injective; it maps at most one fibre to each T_pM and on that fibre it is injective. We claim that this allows us to think of TZ as a subbundle of $TM|_Z$. Indeed, taking $E = TZ$, $X = Z$, $E' = TN$, $X' = M$ and $G = di$ in the previous lemma about images of bundle maps, we see that $\mathrm{im}(di)$

is a subbundle of $TM|_Z$. Of course it is also true that $\ker(di)$ is subbundle of TZ, but it is 0-dimensional. This makes precise the statement that "TZ is a subbundle of $TM|_Z$."

Example 10.1.5. By the Whitney embedding theorem, TM is a subbundle of $T\mathbb{R}^N|_M$, which is the trivial bundle of dimension N over M. We conclude that the tangent bundle to a compact manifold is always a subbundle of a trivial vector bundle.

10.2 Existence of partitions of unity

We now prove the existence of partitions of unity. Before doing so, we must establish a few results about the point-set topology of M, which require that M is second-countable and Hausdorff.

Lemma 10.2.1. *M is a union of countable many open subsets with compact closure.*

Proof. Let $\{W_i\}_{i \in I}$ denote the countable basis for the topology of M and let $\mathcal{A} = \{(U_\alpha, V_\alpha, \phi_\alpha)\}$ be the atlas of M. If there is a codomain V_α of a chart that contains W_i, pick one and call it V_i. This gives a collection of open $\{V_i\}_{i \in I'}$ indexed by a subset $I' \subset I$. We have $\bigcup_{i \in I'} V_i = M$, because the V_α cover M by definition of an atlas and V_α is a union of elements of the basis $\{W_i\}_{i \in I}$ by definition of a basis for a topology.

Given a chart (U_i, V_i, ϕ_i) for $i \in I'$, take all open balls $B_{\epsilon_j}(x_j) \subset U_i$ in its domain such that $\epsilon_j > 0$ is rational, $x_j \in U_i$ has rational coordinate, and the closure $\overline{B}_{\epsilon_j}(x_j)$ is contained in U_i. We denote these

$$W_i^j := \phi_i(B_{\epsilon_j}(x_j)),$$

indexed by some countable set J_i. The collection of all of these is a countable union of countable sets, so is countable. We will prove that $\{W_i^j\}_{i \in I', j \in J_i}$ is the sought-after collection of open subsets.

To see that the W_i^j cover M, we remark that for fixed i we have $\bigcup_{j \in J_i} W_i^j = V_i$ and then varying i we have

$$\bigcup_{i \in I'} \bigcup_{j \in J_i} W_i^j = \bigcup_{i \in I'} V_i = M.$$

The image of the compact set $\overline{B}_{\epsilon_j}(x_j)$ under ϕ_i is compact. Because M is Hausdorff each compact subset is closed and thus the closure of $\phi_i(B_{\epsilon_j}(x_j))$ is contained in $\phi_i(\overline{B}_{\epsilon_j}(x_j))$. Hence it is a closed subset of a compact set, so itself compact. $\square$

Lemma 10.2.2. *There are compact subsets $K_i \subset M$, indexed by integers $i \geqslant 0$, and open subsets $V_{i+1/2} \subset M$ such that $K_0 \subset V_{1/2} \subset K_1 \subset V_{1+1/2} \subset \cdots$ and $\bigcup_{i \geqslant 0} K_i = M$.*

Proof. Let $M = \bigcup_{i\in\mathbb{N}} W_i$ with $\overline{W}_i$ compact. We define the K_i inductively, starting with $K_0 = \overline{W}_0$. Suppose we have defined K_{n-1}, then let N be the smallest integer $\geqslant n$ such that $K_{n-1} \subset W_1 \cup \cdots \cup W_N$. Set $V_{n-1/2} := W_1 \cup \cdots \cup W_N$ and $K_n := \overline{W}_1 \cup \cdots \cup \overline{W}_N$. □

If $\mathcal{U}$ is an open cover of X, we say taht a second open cover $\mathcal{V}$ is a *refinement* if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. One can deduce from the previous lemma that M is *paracompact*, i.e. every open cover has a refinement to a locally finite subcover and it is then a standard fact in point-set topology that partitions of unity by continuous functions exist. We instead want partitions of unity by smooth functions, so we must use somehow that M is a smooth manifold. We first prove a slightly weaker version of Theorem 10.1.2 and along the way we will prove that M is paracompact.

Proposition 10.2.3. *Every open cover $\mathcal{W} = \{W_i\}_{i\in I}$ of M has a refinement which admits a subordinate partition of unity.*

Proof. Let $K_0 \subset V_{1/2} \subset K_1 \subset V_{1+1/2} \subset \cdots$ be as above M and $\mathcal{W} = \{W_i\}_{i\in I}$ be the open cover. Any $p \in M$ lies in a unique $K_n \backslash K_{n-1}$, which has $V_{n+1/2}\backslash K_{n-1}$ as an open neighborhood. We can then pick a chart $(U_\beta, V_\beta, \phi_\beta)$ of M, a point $z \in U_\beta$, and $\delta > 0$, such that $B_\delta(z) \subset U_\beta$, $p = \phi_\beta(z)$ and $\phi_\beta(B_\delta(z)) \subset W_i \cap V_{n+1/2}\backslash K_{n-1}$ for some i.

Ranging over all $p \in M$ (and thus implicitly all $n \geqslant 0$), the open sets $\phi_\beta(B_{\delta/3}(z))$ in particular cover the compact set $K_{m+1}\backslash V_{m-1/2}$, hence there is a finite subcover $\phi_{\beta_i^m}(B_{\delta_i^m/3}(z_i^m))$, $1 \leqslant i \leqslant j_m$ of $K_{m+1}\backslash V_{m-1/2}$. Taking the $\{\phi_{\beta_i^m}(B_{\delta_i^m/3}(z_i^m))\}_{1\leqslant i\leqslant j_m}$ for all m, these give a cover of M, as

$$\bigcup_{m\geqslant 0} K_{m+1}\backslash V_{m-1/2} \supset \bigcup_{m\geqslant 0} K_{m+1}\backslash K_m = M.$$

By construction $\phi_{\beta_i^m}(B_{\delta_i^m/3}(z_i^m))$ is contained in W_i, so this is a refinement of $\mathcal{W}$. It is locally finite since the open subsets $\phi_{\beta_i^m}(B_{\delta_i^m}(z_i^m))$ can only intersect the open subset $V_{n+1/2}\backslash K_{n-1}$ for $n = m-1, m$. *At this point we have proven that M is paracompact.*

In Problem 26 you will show that there exists a smooth function $\tilde{\rho}_i^m \colon U_{\beta_i^m} \to [0,1]$ which vanishes outside $B_{\delta_i^m/2}(z_i^m)$ and is equal to 1 on $B_{\delta_i^m/3}(z_i^m)$. We can then define a smooth map $\tilde{\eta}_i^m \colon M \to [0,1]$ by

$$\tilde{\eta}_i^m(p) = \begin{cases} \tilde{\rho}_i^m(\phi_{\beta_i^m}^{-1}(p)) & \text{if } p \in V_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Since the collection of open subsets $\phi_{\beta_i^m}(B_{\delta_i^m/3}(z_i^m))$ covers M and the collection of open subsets $\phi_{\beta_i^m}(B_{\delta_i^m}(z_i^m))$ is locally finite, we have that

$$p \longmapsto \sum \tilde{\eta}_i^m(p)$$

is locally equal to a finite sum of non-zero terms, so is a smooth map $M \to \mathbb{R}_{>0}$. We then define $\eta_i^m \colon M \to [0,1]$ by

$$\eta_i^m := \frac{\tilde{\eta}_i^m}{\sum \tilde{\eta}_i^m}.$$

This is the desired partition of unity subordinate to the refinement of $\mathcal{W}$ given by the $\phi_{\beta_i^m}(B_{\delta_i^m/3}(z_i^m))$. ☐

Remark 10.2.4. If M is compact, the proof of Theorem 10.1.2 greatly simplifies as you can forget about the K_i and $V_{i+1/2}$'s.

The above construction has multiple functions with support in W_i. Instead, it is often more convenient to have one function for each W_i in $\mathcal{W}$.

Proof of Theorem 10.1.2. By the previous proposition we can find a refinement $\mathcal{W}' = \{W_j\}_{j\in J}$ of $\mathcal{W} = \{W_i\}_{i\in I}$ and a partition of unity $\{\eta_j'\colon M \to [0,1]\}$ subordinate to it.

For $j \in J$, fix a W_i such that $W_j' \subset W_i$. This gives a function $\lambda\colon J \to I$. We claim that

$$\eta_i := \sum_{j\in J^{-1}(i)} \eta_j'$$

gives the desired partition of unity. By property (ii), this is a locally finite sum and hence a smooth function. By property (i), the sum of the η_i is 1 everywhere. From property (i), we know that $\operatorname{supp}(\eta_j') \subset W_j'$ and hence is also contained in W_i. Now observe that

$$\operatorname{supp}(\eta_i) = \overline{\eta_i^{-1}((0,1])} = \overline{\bigcup_{j\in J^{-1}(i)} (\eta_j')^{-1}((0,1])}.$$

By property (ii), the latter is a closure of a locally finite union of open subsets. This is equal to the union of the closures, by an elementary argument in point-set topology. So we conclude that

$$\operatorname{supp}(\eta_i) = \bigcup_{j\in J^{-1}(i)} \overline{(\eta_j')^{-1}((0,1])} = \bigcup_{j\in J^{-1}(i)} \operatorname{supp}(\eta_j') \subset W_i.$$

This finishes the proof. ☐

10.3 Problems

Problem 26 (A bump function)**.**

(a) Prove that

$$\begin{aligned} f\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0 \end{cases} \end{aligned}$$

is smooth.

(b) Observe that $g(x) = f(x)f(1-x)$ is smooth, positive on $(0,1)$, and 0 outside of this interval. Prove that

$$h(x) = \frac{\int_{-\infty}^{x} g(y)dy}{\int_{-\infty}^{\infty} g(y)dy}$$

is smooth, equal to 0 when $x \leqslant 0$ and equal to 1 when $x \geqslant 1$.

(c) Construct a smooth function on $\mathbb{R}^k$ which is 1 on an open neighborhood of the origin and is supported in the unit ball.

Problem 27 (Charts from coordinate axes)**.** Suppose that M is a k-dimensional smooth manifold and $e\colon M \to \mathbb{R}^N$ is a smooth embedding. Prove that for each $p \in M$ there is an open subset $U \subset M$ containing p and integers $i_1, \ldots, i_k$ in $\{1, \ldots, N\}$ such that

$$\begin{aligned} M \supset U &\longrightarrow \mathbb{R}^k \\ p &\longmapsto (\pi_{i_1} \circ e(p), \ldots, \pi_{i_k} \circ e(p)) \end{aligned}$$

is a diffeomorphism onto an open subset. Here $\pi_{i_j} : \mathbb{R}^N \to \mathbb{R}$ is the projection on the i_jth coordinate.

Chapter 11

Transversality and the improved preimage theorem

In this lecture we improve the pre-image theorem to give a sufficient condition under which pre-images of submanifolds are submanifolds. This will have many applications, among them a generalization of the Whitney embedding theorem to non-compact manifolds.

11.1 The preimage theorem restated

Recall that given a submanifold $Z \subset M$, with $i\colon Z \to M$ denoting the inclusion, we have that by considering the image of di we can consider TZ as a subbundle of $TM|_Z$. This makes precise the statement that "TZ is a subbundle of $TM|_Z$."

Many submanifolds arise through the pre-image theorem: we have a smooth map $f\colon M \to N$ with regular value c and $Z = f^{-1}(c)$. The pre-image theorem said that Z is then $(k-k')$-dimensional submanifold of M and $T_p f^{-1}(c) = \ker(d_p f\colon T_pM \to T_{f(p)}M)$ for all $p \in f^{-1}(c)$. The latter part about the tangent spaces to Z, can be improved to a statement about tangent bundles. The proof is identical, but it is only now that we can phrase it:

Theorem 11.1.1 (Preimage theorem)**.** *If $f\colon M \to N$ is a smooth map and $c \in N$ a regular value, then $Z := f^{-1}(c)$ is a $(k-k')$-dimensional submanifold of M and $TZ = \ker(df\colon TM|_Z \to TN) \subset TM|_Z$.*

Example 11.1.2. Recall that S^{n-1} can be written as $g^{-1}(1)$ with $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $(x_1, \ldots, x_n) \mapsto x_1^2 + \ldots + x_n^2$. The map g is smooth and has total derivative $[2x_1, \ldots, 2x_n]$, so all non-zero real numbers are regular values of g. In particular, S^{n-1} is an $(n-1)$-dimensional differentiable manifold and TS^{n-1} is the kernel of the total derivative maps; for $x = (x_1, \ldots, x_n) \in S^{n-1}$ the kernel of $[2x_1, \ldots, 2x_n]$ is just the $(n-1)$-dimensional plane $x^\perp$ of vectors orthogonal to x.

11.2 Transversality

The most important geometric notion in differential topology is transversality. This condition tells you in terms of tangent spaces when submanifolds (or the image of a map and a submanifold) intersect nicely.

11.2.1 Submanifolds locally

We start by recalling the definition of a submanifold, and describe how in suitable local coordinates all submanifolds are the inverse images of projection maps.

Suppose that we have a k'-dimensional differentiable manifold N with a submanifold $Z \subset N$ of codimension r (that is, Z is $(k'-r)$-dimensional). Then for each $z \in Z$ we have a local parametrization, that is, open subsets $U \subset \mathbb{R}^{k'}$ and $V \subset N$ as well as a diffeomorphism $\phi\colon U \to V$ so that $\phi^{-1}(Z \cap V) = U \cap (\{0\} \times \mathbb{R}^{k'-r})$. That is, on U we can define $\pi_r\colon U \to \mathbb{R}^r$ projecting onto the first r coordinates and $\phi^{-1}(Z \cap V) = \pi_r^{-1}(0)$. Thus we see that

$$Z \cap V \subset V = \phi(\pi_r^{-1}(0)) \subset V.$$

If we want to explicit understand $T_zZ \subset T_zN$, then we may as well identify it in U by applying the linear isomorphism $d_z\phi^{-1}$. Here it is the tangent space to $U \cap (\{0\} \times \mathbb{R}^{k'-r})$ at $\phi^{-1}(z)$, which is just $\{0\} \times \mathbb{R}^{k'-r}$. Applying the inverse $d_{\phi^{-1}(z)}\phi$ of $d_z\phi^{-1}$, we see that T_zZ is the following $(k'-r)$-dimensional linear subspace of T_zN:

$$T_zZ = d_{\phi^{-1}(z)}\phi(\{0\} \times \mathbb{R}^{k'-r}) \subset T_zN. \tag{11.1}$$

11.2.2 Improving the pre-image theorem

Now suppose we have a smooth map $f\colon M \to N$. We will give a criterion that tells us when $f^{-1}(Z)$ is a differentiable submanifold of M.

To find a local parametrization of $f^{-1}(Z) \subset M$ near $p \in f^{-1}(Z)$, we might as well find one of $f^{-1}(Z \cap V) \subset f^{-1}(V) \subset M$. The advantage of passing to this open subset is that on $f^{-1}(V)$ we can use projection to define the smooth map

$$g \coloneqq f^{-1}(V) \subset M \xrightarrow{f} V \subset N \xrightarrow{\phi^{-1}} U \subset \mathbb{R}^{k'} \xrightarrow{\pi_r} \mathbb{R}^r.$$

This has the property that

$$g^{-1}(0) = f^{-1}(\phi(\pi_r^{-1}(0))) = f^{-1}(\phi(\phi^{-1}(Z \cap V))) = f^{-1}(Z \cap V).$$

The pre-image theorem then tells us that $f^{-1}(Z \cap V)$ is a submanifold of $f^{-1}(V) \subset M$ of codimension r *whenever* 0 *is regular value of* g. That is, g should be a submersion at all $p \in f^{-1}(Z \cap V)$.

So we need to understand when $d_pg\colon T_pM \to T_0\mathbb{R}^r$ is surjective. Writing

$$d_pg = d_{\phi^{-1}f(p)}\pi_r \circ d_{f(p)}\phi^{-1} \circ d_pf,$$

we first observe that for d_pg to be surjective, $\mathrm{im}(d_{f(p)}\phi^{-1} \circ d_pf)$ should be a linear subspace of $T_{\phi^{-1}f(p)}\mathbb{R}^{k'} = \mathbb{R}^{k'}$ which surjects onto $T_0\mathbb{R}^r = \mathbb{R}^r$ under the linear map $d_{\phi^{-1}f(p)}\pi_r\colon \mathbb{R}^{k'} \to \mathbb{R}^r$. This is the case exactly when $\mathrm{im}(d_{f(p)}\phi^{-1} \circ$

$d_pf) + \ker(d_{\phi^{-1}f(p)}\pi_r) = \mathbb{R}^{k'}$. Using the fact that $\ker(d_{\phi^{-1}f(p)}\pi_r) = \{0\} \times \mathbb{R}^{k'-r}$ we obtain the requirement

$$\mathrm{im}(d_{f(p)}\phi^{-1} \circ d_pf) + \{0\} \times \mathbb{R}^{k'-r} = \mathbb{R}^{k'}.$$

Let us apply the linear isomorphism $d_{\phi^{-1}f(p)}\phi$ to translate this back to a statement about linear subspaces of the original tangent space $T_{f(p)}N$. By the chain rule $d_{\phi^{-1}f(p)}\phi$ sends $\mathrm{im}(d_{f(p)}\phi^{-1} \circ d_pf)$ to $\mathrm{im}(d_pf)$, and by (11.1) it sends $\{0\} \times \mathbb{R}^{k'-r}$ to $T_{f(p)}Z$. Since a linear isomorphism preserves sums, we see that d_pg is surjective if and only if

$$\mathrm{im}(d_pf) + T_{f(p)}Z = T_{f(p)}N.$$

Let us give this condition a name:

Definition 11.2.1. Let $Z \subset N$ be a submanifold. We say that $f\colon M \to N$ is *transverse to Z at $p \in f^{-1}(Z)$*, denoted $f \pitchfork_p Z$, when $\mathrm{im}(d_pf) + T_{f(p)}Z = T_{f(p)}N$.

Definition 11.2.2. Let $Z \subset N$ be a submanifold. We say that $f\colon M \to N$ is *transverse to Z* , denoted $f \pitchfork Z$, when f is transverse to Z at all $p \in f^{-1}(Z)$.

Example 11.2.3. A smooth map $f\colon \mathbb{R} \to \mathbb{R}^2$ is transverse to $\mathbb{R} \times \{0\}$ if and only if the derivative $\partial f_2/\partial t$ is non-zero whenever $f(t)$ crosses the x-axis.

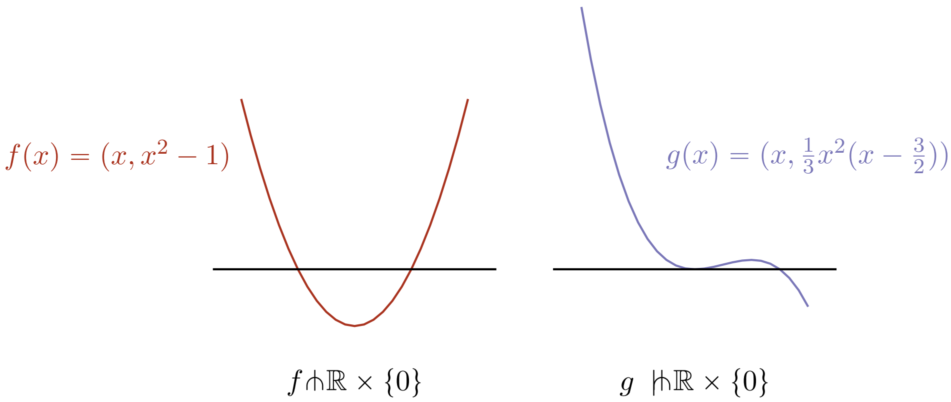


Figure 11.1 Examples of smooth functions $\mathbb{R} \to \mathbb{R}^2$.

Then the above discussion tells us that $f\colon M \to N$ being transverse to Z at all $p \in f^{-1}(Z \cap V)$ implies that $f^{-1}(Z \cap V) = g^{-1}(0)$ is a submanifold. Varying the local parametrizations, we see that f being transverse to Z implies $f^{-1}(Z)$ is a submanifold. We can say a bit more; by the pre-image theorem the tangent space to $f^{-1}(Z \cap V) = g^{-1}(0)$ at p is given by the kernel of d_pg, i.e. $(d_pg)^{-1}(0)$, which is equal to $(d_pf)^{-1}(T_{f(p)}Z)$.

Theorem 11.2.4 (Improved preimage theorem)**.** *Let $Z \subset N$ a submanifold of codimension r and suppose that $f\colon M \to N$ that is transverse to Z. Then $f^{-1}(Z) \subset M$ is also a submanifold of codimension r and $Tf^{-1}(Z) = (df)^{-1}(TZ) \subset TM|_{f^{-1}(Z)}$.*

Remark 11.2.5. This is an improvement of the preimage theorem, because we can recover the preimage theorem by take Z to be a point c. Since the tangent space to the (rather boring) 0-dimensional manifolds c is 0-dimensional, f is transverse to c at $p \in f^{-1}(c)$ if and only if $d_p f$ is surjective.

Example 11.2.6. Suppose $Z \subset N$ is a collection of points and M is of smaller dimension than N. Then $f\colon M \to N$ is transverse to Z if and only $\mathrm{im}(f) \cap Z = \varnothing$, as it is not possible for the sum of a 0-dimensional and $< k'$-dimensional subspace to equal a k'-dimensional vector space.

Example 11.2.7. Though $f \pitchfork Z$ implies that $f^{-1}(Z)$ is a submanifold, the converse is not true: the inclusion $i\colon Z \to N$ is very much not transverse to Z, but $i^{-1}(Z) = Z$.

Remark 11.2.8. You may want to try to come up with the definition of two smooth maps $f\colon M_1 \to N$ and $g\colon M_2 \to N$ being transverse, and then prove that $\{(m_1, m_2) \mid f(m_1) = g(m_2)\} \subset M_1 \times M_2$ is a submanifold.

11.2.3 Transversality for submanifolds

The case that is of most geometric interest is when f is the inclusion $j\colon Y \to N$ of another submanifold. In that case, it is more convenient to forget about the maps $i\colon Z \to N$ and $j\colon Y \to N$ and state the transversality condition in terms of the submanifolds:

Definition 11.2.9. Let $Y, Z \subset N$ be submanifolds. Then Y and Z are *transverse at* $p \in Y \cap Z$, denoted $Y \pitchfork_p Z$, if $T_pY + T_pZ = T_pN$.

Definition 11.2.10. Let $Y, Z \subset N$ be submanifolds. Then Y and Z are *transverse*, denoted $Y \pitchfork Z$, if Y and Z are transverse at all $p \in Y \cap Z$.

Example 11.2.11. If $Y \cap Z = \varnothing$, $Y \pitchfork Z$ because there are no points in $p \in Y \cap Z$ at which any conditions are imposed.

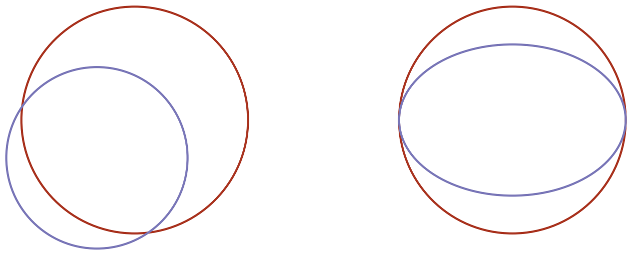
The improved pre-image theorem says that if $Y \pitchfork Z$ then $Y \cap Z$ is a submanifold of Y, and hence a submanifold of N. (If this sounds surprising, you should go through the definitions again and verify that a submanifold of a submanifold is a submanifold). At each $p \in Y \cap Z$, $T_p(Y \cap Z) = T_pY \cap T_pZ$. This in particular implies that

$$\mathrm{codim}(Y \cap Z) = \mathrm{codim}(Y) + \mathrm{codim}(Y).$$

You should think of $Y \pitchfork Z$ as saying that Y and Z intersect nicely. Let us make this more precise:

Example 11.2.12. Two linear subspaces U and V in $\mathbb{R}^n$ of codimension r and s respectively intersect transversally if and only if $U \cap V$ is a linear subspace of codimension $r + s$.

The direction $\Rightarrow$ is a consequence of the general formula for the codimension of a transverse intersection. For the direction $\Leftarrow$, we note that at each $p \in U \cap V$ we can identity T_pU and T_pV with U and V again. To compute their sum $U + V$



transverse not transverse

Figure 11.2 Examples of 1-dimensional submanifolds of $\mathbb{R}^2$.

we use the inclusion-exclusion formula for the dimension of a sum of two linear subspaces:

$$\dim(U+V) = \dim(U)+\dim(V)-\dim(U+V) = (n-r)+(n-s)-(n-r-s) = n.$$

Hence $U+V = \mathbb{R}^n$ and U and V intersect transversally at $p \in U \cap V$.

Any transverse intersection locally is of the form in Example 11.2.12 in the right coordinates:

Lemma 11.2.13. *$Y \pitchfork_p Z$ if and only if there is a chart $(U_\alpha, V_\alpha, \phi_\alpha)$ such that $\phi_\alpha^{-1}(Y)$ and $\phi_\alpha^{-1}(Z)$ are given by the intersection with U_α of two linear subspaces intersecting transversally.*

Proof. $\Leftarrow$ follows from transversality being preserved by diffeomorphisms, so we focus on $\Rightarrow$. Since the intersection is non-empty the codimensions r and s of Y and Z satisfiy $r + s \leqslant k'$.

The proof of the improved pre-image theorem provides a chart (U_1, V_1, ϕ_1) in which $\phi^{-1}(Z) = U_1 \cap (\{0\} \times \mathbb{R}^{k'-r})$. We may assume that $\phi_1(0) = p$ by translating. Translated to this chart, $Y \pitchfork_p Z$ says that $T_0\phi_1^{-1}(Y) + \{0\} \times \mathbb{R}^{k'-r} = \mathbb{R}^{k'}$. Thus by applying a linear isomorphism of $\mathbb{R}^{k'}$ preserving $\{0\} \times \mathbb{R}^{k'-r}$ we may assume that $T_0\phi_1^{-1}(Y) = \mathbb{R}^{k'-s} \times \{0\}$.

So it remains to fix $\phi_1^{-1}(Y)$. Consider the map $\pi\colon \phi_1^{-1}(Y) \to \mathbb{R}^{k'-s} \times \{0\}$ given by restricting the projection map $\mathbb{R}^{k'} \to \mathbb{R}^{k'-s} \times \{0\}$. The derivative of π at 0 is the identity and hence bijective. Inverse function theorem then tells us that π is a local diffeomorphism. Thus near the origin,

$$\phi_1^{-1}(Y) = \{(w, \rho(w)) \in \mathbb{R}^{k'-s} \times \mathbb{R}^s\}$$

for a smooth map $\rho\colon \mathbb{R}^{k'-s} \to \mathbb{R}^s$ with $\rho(0) = 0$. Thus there exists an open subset U_2 of the origin in $\mathbb{R}^{k'}$ so that the diffeomorphism $\bar{\rho}\colon \mathbb{R}^{k'} \to \mathbb{R}^{k'}$ given by $\bar{\rho}(w, v) = (w, v + \rho(w))$ maps $U_2 \cap (\mathbb{R}^{k'-s} \times \{0\})$ onto a neighborhood of the origin in $\phi_1^{-1}(Y)$. Note that $\bar{\rho}$ preserves $\{0\} \times \mathbb{R}^{k-r}$, we only translate in the last s coordinates and $s \leqslant k' - r$ as $k' \geqslant r + s$. Thus the desired chart is

$$(U_2, V_2, \phi_2) := (U_2, \phi_1 \circ \bar{\rho}(V_2), \phi_1 \circ \bar{\rho}). \qquad \square$$

11.3 Another construction of the Poincaré homology sphere

As an extended example, we will now give an alternative and at first sight completely unrelated construction of the Poincaré homology sphere $P = S^3/I^*$, which we first saw as an additional example.

To do so, we consider the map

$$f\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$$
$$(z_1, z_2, z_3) \longmapsto z_1^2 + z_2^3 + z_3^5.$$

We claim that

$$X = f^{-1}(0) \cap (\mathbb{C}^3 \backslash 0)$$

is a codimension 2 submanifold of $\mathbb{C}^3\backslash\{0\}$. We of course would like to use the submersion theorem, and we could do by identifying the domain $\mathbb{C}^3$ with $\mathbb{R}^6$ by $z_j \longleftrightarrow x_j + iy_j$, and similarly identify the target $\mathbb{C}$ with $\mathbb{R}^2$. We would then need to verify that the total derivative, a (2×6)-matrix, is surjective.

However, it is much convenient to keep working with complex numbers: as a polynomial, p is not only differentiable as a function $\mathbb{R}^6 \to \mathbb{R}^2$, is in fact complex-differentiable as a function $\mathbb{C}^3 \to \mathbb{C}$. We can compile these into a (1×3)-matrix of complex numbers

$$\begin{bmatrix} \frac{\partial f}{\partial z_1}(z_1, z_2, z_3) & \frac{\partial f}{\partial z_2}(z_1, z_2, z_3) & \frac{\partial f}{\partial z_3}(z_1, z_2, z_3) \end{bmatrix}.$$

This complex total derivative is surjective if and only if the total derivative is surjective.

In our case, the complex total derivative is given by

$$\begin{bmatrix} 2z_1 & 3z_2^2 & 5z_3^4 \end{bmatrix} \tag{11.2}$$

and hence surjective for all $(z_1, z_2, z_3) \in \mathbb{C}^3\backslash\{0\}$. We conclude that $X \subset \mathbb{C}^3\backslash\{0\}$ is a 4-dimensional smooth manifold, or equivalently has codimension 2.

To reduce the dimension by one, we will intersect with the sphere $S^5 := \{(z_1, z_2, z_3) \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$, of codimension 1. We claim this is transverse to X. To see this is the case, we use that the tangent bundle to X at $x = (z_1, z_2, z_3) \in X$ is given by the kernel of the matrix (11.2); this has fibers isomorphic to $\mathbb{C}^2 \cong \mathbb{R}^4$ so is 4-dimensional. A particular vector in this kernel is

$$w = (z_1/2, z_2/3, z_3/5).$$

The tangent bundle to S^5 at $x \in S^5$ is given by those vectors orthogonal to x; this is 5-dimensional. It is convenient to work with complex numbers, and observe that $w = (w_1, w_2, w_3) \in T_x\mathbb{C}^3$ being orthogonal to x is equivalent to

$$\mathrm{Re}(\overline{x} \cdot w) = 0.$$

Let us evaluate this on the above vector in T_xX: we get

$$\mathrm{Re}(\overline{x} \cdot w) = |z_1|^2/2 + |z_2|^2/3 + |z_3|^2/5,$$

and since $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, we see this is at least $1/5$ so non-zero. Thus $T_x X \not\subset T_x S^5$ and by a dimension count we conclude that $T_x X + T_x S^5 = T_x \mathbb{C}^3$. Thus $f^{-1}(0) \cap S^5$ is a submanifold of S^5. Its codimension is $2 + 1$, so it is 3-dimensional. It is in fact diffeomorphic to the Poincaré homology sphere [KS79, p. 128–132].

Remark 11.3.1. A smooth manifold which arises as the transverse intersection of a zero set of a complex polynomial (here $z_1^2 + z_2^3 + z_3^5$) with a small sphere around a singularity (here we took the sphere of radius one around the origin), is called a *link of a singularity.* These have been studied in detail, see e.g. [Mil68]. Particularly interesting are the *Brieskorn spheres* $\Sigma(k_1, \ldots, k_n)$, constructed as the links of the singularity at the origin of polynomials

$$z_1^{k_1} + \cdots + z_n^{k_n}.$$

These give examples of smooth manifolds which are homeomorphic of spheres but not diffeomorphic to them [Bri66]. For examples, the cases

$$\Sigma(2, 2, 2, 3, 6k - 1)$$

give all examples of exotic 7-spheres up to diffeomorphism.

11.4 Problems

Problem 28 (Brieskorn manifolds)**.** Verify that all Brieskorn spheres are $(2n-3)$-dimensional manifolds.

Problem 29 ($\mathbb{R}P^3$ as a link of a singularity)**.** Recall the smooth manifold W_n from a previous problem. Use the map

$$\begin{aligned} f\colon \mathbb{C}^2 &\longrightarrow \mathbb{C}^3 \\ (w_1, w_2) &\longmapsto (w_1^2 + w_2^2,\ i(w_1^2 - w_2^2),\ 2iw_1w_2) \end{aligned}$$

to produce a diffeomorphism $\mathbb{R}P^3 \to W_3$. Conclude that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}P^3$.

Chapter 12

Stable and generic classes of smooth maps

It is a standard strategy to study the effect of small deformations on mathematical objects. On the one hand, such deformations can make the object more generic and hence easier to understand. On the other hand, small enough deformations often preserve important properties. To start applying this strategy to certain types of smooth maps, we will need to do the following:

(i) make precise what we mean by a "deformation,"

(ii) understand which types of smooth maps are "stable", i.e. preserved by small deformations, and

(iii) understand what a "generic smooth map" looks like.

12.1 Homotopies of smooth maps

A reasonable definition of deforming of a smooth map f_0 is to situate it in a family of smooth maps f_s which depends smoothly on the parameter s. Restricting the parameter s to lie in the closed interval $[0, 1]$, we get the following definition:

Definition 12.1.1. A *homotopy* is a smooth map $H\colon M \times [0, 1] \to N$.

Example 12.1.2. Out of a smooth map $f\colon M \to N$, we can construct a constant homotopy $H\colon M \times [0, 1] \to N$ by $H(p, t) := f(p)$. This homotopy does not deform f at all!

We have not officially said what it means to have a smooth map with domain $M \times [0, 1]$; we will later define manifolds with boundary, but for now it suffices to say that it should extend to a smooth map whose domain is an open neighbourhood of $M \times [0, 1]$ in $M \times \mathbb{R}$.

Since the restrictions of smooth maps are smooth, each $f|_{M\times\{t\}}\colon M \to N$ is a smooth map. In particular this is the case for $f_0 := f|_{M\times\{0\}}$ and $f_1 := f|_{M\times\{1\}}$ and we say that H is a homotopy from f_0 to f_1.

Definition 12.1.3. Two smooth maps $f_0, f_1\colon M \to N$ are *homotopic*, denoted $f_0 \sim f_1$, if there is a homotopy from f_0 to f_1.

Lemma 12.1.4. *Homotopy is an equivalence relation of smooth maps $M \to N$.*

Proof. The constant homotopy shows it is reflexive. To see it is symmetric, note that if $H\colon M \times [0,1] \to N$ is a homotopy from f_0 to f_1 then $\bar{H}(p,t) := H(p, 1-t)$ is a homotopy from f_1 to f_0. In Problem 30 you will show it is transitive. □

12.2 Stable classes of maps

A class of smooth maps is stable if it is preserved by small perturbations, in the following sense:

Definition 12.2.1. A subset U of the set of all smooth maps $M \to N$ is *stable* if for each $f_0 \in U$ and smooth map $H\colon M \times \mathbb{R}^r \to N$ starting at f_0 there exists an $\epsilon > 0$ such that $H|_{M\times\{x\}} \in U$ for all $||x|| < \epsilon$.

This definition has a straightforward consequence for homotopies:

Lemma 12.2.2. *If U is stable then for each $f_0 \in U$ and homotopy $H\colon M \times [0,1] \to N$ starting at f_0 there exists an $\epsilon > 0$ such that $H|_{M\times\{t\}} \in U$ for all $t < \epsilon$.*

Proof. There exists a smooth map $\eta\colon \mathbb{R} \to [0,1]$ such that $\eta(t) = 0$ for $t \leqslant 0$ and $\eta'(t) > 0$ for $t > 0$. Now apply the condition in the definition of stable classes of maps to $H \circ (\mathrm{id} \times \eta)\colon M \times \mathbb{R} \to N$. □

Remark 12.2.3. If we were to go to the trouble of defining a suitable topology on the set $C^\infty(M,N)$ of smooth maps $M \to N$, open subsets of $C^\infty(M,N)$ would be stable.

This remark makes us suspect that subsets which are defined by "open conditions" should be stable. Let us look at an example: in the space $\mathrm{Lin}(\mathbb{R}^p, \mathbb{R}^p)$ of all linear maps $\mathbb{R}^p \to \mathbb{R}^p$ the invertible linear maps are open (as they are defined by the condition that the determinant is non-zero). This means that if an invertible $A \in \mathrm{Lin}(\mathbb{R}^p, \mathbb{R}^p)$ is perturbed slightly, it remains invertible. Since a map $f\colon M \to N$ is a local diffeomorphism if and only if all derivatives d_pf are invertible, one might expect that this condition should be preserved by a small perturbation of f, as it gives rise to a small perturbation of each d_pf. Thus, if we could somehow "bound the determinant of the d_pf" away from 0, any small perturbation of f will remain a local diffeomorphism.

The problem with this vague argument is of course that one can't make sense of the determinant of a linear map between two different vector spaces. The idea is to use the determinant in finitely many charts, and to guarantee M is covered by finitely many charts we assume it is compact.

Let us now make it precise:

Theorem 12.2.4. *If M is compact, then the following classes of smooth maps $f\colon M \to N$ are stable:*

(i) local diffeomorphisms,

(ii) immersions,

(iii) submersions,

(iv) maps transverse to a submanifold $Z \subset N$,

(v) embeddings,

(vi) diffeomorphisms.

Proof. The case (i) is a special case of both (ii) and (iii). Case (ii) is very similar to case (iii) and proven in Guillemin & Pollack, so we will only prove the latter. Suppose $f_0 \colon M \to N$ is a submersion and $H \colon M \times \mathbb{R}^r \to N$ is a smooth map so that $H|_{M\times\{0\}} = f_0$. Pick a finite collection of charts $\{(U_i, V_i, \phi_i)\}$, $1 \leqslant i \leqslant r$, such that $\bigcup_i V_i = M$ and $f(V_i) \subset V'_{j(i)}$ for some chart $(U'_{j(i)}, V'_{j(i)}, \phi'_{j(i)})$ of N. Taking a partition of unity $\eta_i \colon M \to [0,1]$, we find compact subsets $K_i := \operatorname{supp}(\eta_i) \subset V_i$ which also cover M. Each compact subset $f_0(K_i)$ is contained in an open subset $V'_{j(i)}$. Hence there exists a $\delta_i > 0$ such that

$$H(K_i \times \overline{B}_{\delta_i}(0)) \subset V'_{j(i)}.$$

For suppose no such $\delta_i > 0$ exists, then there is a sequence (p_k, t_k) with $p_k \in K_i$, $t_k \to 0$ and $H(p_k, t_k) \in N \backslash V'_{j(i)}$. Since M is compact, without loss of generality p_k converges to p. Since $N \backslash V'_{j(i)}$ is closed, we get

$$N \backslash V'_{j(i)} \ni \lim_k H(p_k, t_k) = H(p, 0) = f_0(p)$$

and thus a contradiction to $f_0(K_i) \subset V'_{j(i)}$. So if we take $\delta = \min(\delta_i \mid 1 \leqslant i \leqslant r) > 0$ we have that $H(K_i \times \overline{B}_\delta(0)) \subset V'_{j(i)}$ for all $1 \leqslant i \leqslant r$.

This setup has the following goal: whether there is an $\epsilon \in (0, \delta)$ such that $H|_{M\times\{t\}}$ for all $||t|| < \epsilon$ is a submersion is equivalent to whether each of the finitely many functions

$$f_t^i := (\phi'_{j(i)})^{-1} \circ H|_{K_i \times \{t\}} \circ \phi_i$$

has a surjective total differential at all points in its domain for all $||t|| < \epsilon$.

Each f_t^i is a smooth map from the compact subset $\phi_i^{-1}(K_i) \subset \mathbb{R}^k$ to the open subset $V'_{j(i)} \subset \mathbb{R}^{k'}$. Consider now the continuous function

$$\phi_i^{-1}(K_i) \times \overline{B}_\delta(0) \ni (p, t) \longmapsto \begin{array}{c}\text{maximum of absolute value of determinants} \\ \text{of } (k' \times k')\text{-submatrices of } d_p f_t^i\end{array}.$$

The right hand side is positive if and only if there is a square submatrix of full rank, which happens if and only if it is surjective. Hence we know that for $t = 0$, the total derivatives at all $x \in \phi_i^{-1}(K_i)$ are surjective and hence the above function is strictly positive. Since $\phi_i^{-1}(K_i)$ is compact, it is bounded away from 0 for $t = 0$, and by continuity thus for all t in some small ball $\overline{B}_{\epsilon_i}(0) \subset \overline{B}_\delta(0)$ with $\epsilon_i > 0$. The argument this is similar to the above argument that $H(K_i \times \overline{B}_\delta(0)) \subset V'_{j(i)}$ for some $\delta_i > 0$, and you should work it out yourself. Taking $\epsilon = \min(\epsilon_i \mid 1 \leqslant i \leqslant r)$ gives the desired $\epsilon > 0$.

We may reduce the case (iv) to the case (iii) by picking finitely many local parametrizations covering the intersection of Z with an open neighbourhood of $f_0(M)$. In the coordinates coming from each of these local parametrizations, Z

is given by $\{0\} \times \mathbb{R}^r$ and by composing with the projection $\pi_{k'-r}$ onto the first $k' - r$ coordinates we can rephrase $f \pitchfork Z$ in terms of $\pi_{k'-r} \circ f$ being a submersion.

For (vi) we may reduce to the case that M and N are connected by considering each connected component separately. But an embedding $f\colon M \to N$ between compact connected manifolds of the same dimension is the same as diffeomorphism. Hence (vi) reduces to (v).

Furthermore, (v) reduces to (ii) as soon as we prove that there must exist an $\epsilon > 0$ such that each $H|_{M\times\{t\}}$ is injective for $t < \epsilon$. Suppose this is not the case, then we will derive a contradiction. Then if we define $\tilde{H}\colon M \times \mathbb{R}^p \to N \times \mathbb{R}^r$ by $\tilde{H}(p,t) = (H(p,t),t)$ we can find a collection of pairs $(r_i,t_i),(p_i',t_i) \in M \times \mathbb{R}^r$ with $t_i \to 0$, $p_i \neq p_i'$ and $\tilde{H}(p_i,t_i) = \tilde{H}(p_i',t_i)$. Using the fact that M is compact, by passing to a subsequence we can assume that both sequences p_i and p_i' converge to p and p' in M. Then $f_0(p) = \lim H(p_i,t_i) = \lim H(p_i',t_i) = f_0(p')$ and since f_0 is injective $p = p'$. We may compute that

$$d_{(p,0)}\tilde{H} = \begin{bmatrix} d_p f_0 & * \\ 0 & \mathrm{id} \end{bmatrix} : T_pM \oplus \mathbb{R}^r \to T_{f_0(p)}N \oplus \mathbb{R}^r,$$

which is injective. Hence $\tilde{H}$ is an embedding near $(p,0)$, so in particular injective and hence $(p_i,t_i) = (p_i',t_i)$ for i large enough, contradicting the construction of the sequences p_i and p_i'. $\square$

Example 12.2.5. If Z is a compact submanifold of M, then any sufficiently small perturbation of the inclusion map $i\colon Z \hookrightarrow M$ is still an embedding. Concretely, when you pick *any* smooth function $g\colon S^1 \to \mathbb{R}^2$, there exists some $\epsilon > 0$ such that

$$\begin{aligned} i_t\colon S^1 &\longrightarrow \mathbb{R}^2 \\ p &\longmapsto p + tg(p) \end{aligned}$$

is an embedding for $t < \epsilon$.

12.3 Generic classes of smooth maps

A class of smooth maps is generic if we can deform any smooth map to such a map by an arbitrarily small perturbation. It will be technically convenient to allow these perturbations to be indexed by $\mathbb{R}^r$ instead of $\mathbb{R}$.

Definition 12.3.1. A subset D of the set of all smooth maps $M \to N$ is *generic* if for all $f_0\colon M \to N$ there exists an $r \geqslant 0$ and a smooth map $H\colon M \times \mathbb{R}^r \to N$ such that $M|_{M\times\{0\}} = f_0$ and for all $\epsilon > 0$ there exists an $x \in \mathbb{R}^r$ with $||x|| < \epsilon$ such that $H|_{M\times\{x\}} \in D$.

Remark 12.3.2. If we were to define a suitable topology on the set $C^\infty(M,N)$ of smooth maps $M \to N$, dense open subsets of $C^\infty(M,N)$ would be generic.

Example 12.3.3. We will later prove that if the set of all smooth maps $M \to N$ transverse to Z is generic. Thus every smooth map $f\colon M \to N$ can be approximated by maps transverse to Z.

The main tool to find generic classes of smooth maps is Sard's theorem, often applied to homotopies or families of maps but incredibly useful in general:

Theorem 12.3.4 (Sard)**.** *If $f\colon M \to N$ is a smooth map, then the critical values of f have measure zero.*

We need to explain the statement: a subset C of $\mathbb{R}^p$ has *measure zero* if there is a countable collection of rectangles $R_i \subset \mathbb{R}^p$ such that $C \subset \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \mathrm{vol}(R_i) < \epsilon$, and a subset C of M has *measure zero* if for each chart $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ of M the subset $\phi_\alpha^{-1}(C)$ has measure zero.

Corollary 12.3.5. *If $f_i\colon M \to N$ is a countable number of smooth maps, then the set of $c \in N$ which are regular values for all f_i is dense.*

Proof. The countable union $\bigcup_i \mathrm{crit}(f_i) \subset N$ of measure zero subsets has measure zero, so it suffices to observe that the complement of a measure zero subset C is dense. If it were not dense, C would have non-empty interior and in some chart contain a small ball of some definite volume > 0. □

Let us give some first applications of Sard's theorem:

Example 12.3.6. There are space-filling curves, continuous maps $f\colon [0,1] \to [0,1]^2$ which are surjective. However, no smooth space-filling curve can exist: a regular value of such a smooth map is a point in $[0,1]^2$ which is not in the image of f, and the regular values need to be dense in $[0,1]^2$ by Sard's lemma.

The following is an elaboration of that idea:

Definition 12.3.7. A path-connected differentiable manifold M is said be to *m-connected* if every smooth map $f\colon S^i \to M$ is homotopic to a constant map for $i \leqslant m$.

Remark 12.3.8. To connect this definition to a more familiar one in algebraic topology involving continuous maps instead of smooth maps, one uses the fact that every continuous map is homotopic to a smooth one.

Corollary 12.3.9. *The sphere S^k is $(k-1)$-connected.*

Proof. As before, the regular values of smooth map $f\colon S^i \to S^k$ for $i \leqslant k-1$ are those that are not in the image of f. Since these must be dense f must miss some point $x_0 \in S^k$. We can then identify $S^k \backslash \{x_0\}$ with $\mathbb{R}^k$ and consider f as a smooth map $f\colon S^i \to \mathbb{R}^k$. This is homotopic to a constant map by the homotopy $H\colon S^i \times [0,1] \to \mathbb{R}^k$ given by $H(p,t) = tf(p)$. □

Next chapter we will use Sard's lemma to improve the Whitney embedding theorem.

12.4 The proof of Sard's theorem

The following is the proof of Sard's theorem, Theorem 12.3.4, which is essentially a result in multivariable calculus and as such not part of the course proper. Its proof is the standard one, and is included for completeness. It needs one fact regarding sets of measure 0, a special case of Fubini's theorem. This falls within the realm of measure theory, so we will assert it without proof (but see Appendix 1 of [GP10] in the case C is closed).

Lemma 12.4.1. *Suppose that we are given an open subset $U \subset \mathbb{R}^{k+1}$ and a subset $C \subset U$ such that $C \cap (\{t\} \times \mathbb{R}^k)$ has measure 0 for all $t \in \mathbb{R}$. Then C has measure 0.*

Theorem 12.4.2. *The set of critical values of any smooth map $f \colon M \to N$ has measure 0.*

Proof. When we proved that partitions of unity exist, we prove that there exists a countable collection of charts $\{(U_i, V_i, \phi_i)\}$ covering M and a countable collection of charts $\{(U'_{i(j)}, V'_{i(j)}, \phi'_{i(j)})\}$ covering N such that $f(V_i) \subset V'_{i(j)}$. The set $\mathrm{Crit}(f)$ of critical values of f is equal to

$$\mathrm{Crit}(f) = \bigcup_i \phi'_{j(i)} \left(\mathrm{Crit}((\phi'_{j(i)})^{-1} \circ f \circ \phi_i)\right).$$

We observed in the proof of Corollary 12.3.5 that subsets of measure 0 are closed under taking countable unions, so it suffices to prove Sard's theorem for each of the functions on the right hand side. That is, it suffices to prove Sard's theorem for smooth maps $f \colon U \to \mathbb{R}^{k'}$ with $U \subset \mathbb{R}^k$ open. We will prove this by induction over k.

In the case $k = 0$, there are either no critical values (when $k' = 0$) or a single one (when $k' > 0$), so this initial case is true. For the induction step from $k-1$ to k, we let $C \subset U$ denote the set of *critical points* of f and filter it by

$$C \supset C_1 \supset C_2 \supset \cdots,$$

letting C_i be the subset where all partial derivatives of order $1 \leqslant r \leqslant i$ vanish. Now we will write C as $(C \backslash C_1) \cup \bigcup_{i \geqslant 1} C_i$. We have to prove $f(C)$ has measure 0. As $f(C) = f(C \backslash C_1) \cup \bigcup_{i \geqslant 1} f(C_i)$ it suffices to prove that $f(C \backslash C_1)$ and $f(C_i)$ for $i \geqslant 1$ have measure 0.

This is done in three steps:

The case $f(C \backslash C_1)$. If $k' = 1$ then $C = C_1$ and there is nothing to prove, so assume $k' \geqslant 2$. At $c \in C \backslash C_1$, $\frac{\partial f_i}{\partial x_j}(c) \neq 0$ for some i and j. Without loss of generality (reordering the coordinate directions) we may assume $i = 1$ and $j = 1$. Define a smooth map

$$\begin{aligned} h \colon U &\longrightarrow \mathbb{R}^k \\ (x_1, \ldots, x_k) &\longmapsto (f_1(x), x_2, \ldots, x_k), \end{aligned}$$

which is easily seen to have bijective total derivative at c. Applying the inverse function theorem, we see it is a local diffeomorphism, i.e. there is an open neighborhood V around c such that h restricts to a diffeomorphism $U \supset V \to h(V) \subset \mathbb{R}^{k'}$.

Now consider the composition of its inverse with f

$$\begin{aligned} f \circ h^{-1} \colon h(V) &\longrightarrow \mathbb{R}^{k'} \\ (x_1, \ldots, x_k) &\longmapsto (x_1, f_2(h(x)), \ldots, f_n(h(x))). \end{aligned}$$

This sends the manifold $h(V) \cap (\{t\} \times \mathbb{R}^{k-1})$ to $\{t\} \times \mathbb{R}^{k'-1}$, and a point (t, c') is a critical point of f if and only if c' is a critical point of

$$\bar{f}_t := (f_2(t, -), \ldots, f_n(t, -)) \colon h(V) \cap (\{t\} \times \mathbb{R}^{k-1}) \longrightarrow \{t\} \times \mathbb{R}^{k'-1}.$$

Applying the inductive hypothesis to each of these, we see that the set of critical values of $\bar{f}_t$ has measure zero.

Letting $C(\bar{f}_t)$ denote the critical points of $\bar{f}_t$, the application of Fubini's theorem discussed above then tells us that

$$\bigcup_t \{t\} \times \bar{f}_t(C(\bar{f}_t))$$

also has measure 0. But that union is exactly the subset of the critical values of $g \circ h^{-1}$ where not all first order partial derivatives vanish. Since h^{-1} is a diffeomorphism, these are also the subset of such critical values of $g|_V$. Thus $f((C \backslash C_1) \cap V)$ has measure 0. Since a countable collection of V's cover $C \backslash C_1$ (using second countability of M), we conclude that $f(C \backslash C_1)$ has measure 0.

The case $f(C_i \backslash C_{i+1})$**.** Starting as in the previous case, at $c \in C_i \backslash C_{i+1}$ we know that $\frac{\partial^{i+1} f_j}{\partial x_{k_1} \cdots \partial x_{k_{i+1}}} \neq 0$ for some j and k_1, and without loss of generality we can assume both are equal to 1. Then we define

$$\begin{aligned} h \colon U &\longrightarrow \mathbb{R}^k \\ (x_1, \ldots, x_k) &\longmapsto \left(\frac{\partial^i f_1}{\partial x_{k_2} \cdots \partial x_{k_{i+1}}}, x_2, \ldots, x_k \right). \end{aligned}$$

As before, h is a diffeomorphism onto its image when restricted to an open neighborhood V of c. It also maps C_i into $\{0\} \times \mathbb{R}^{k-1}$, because the first entry involves an ith partial derivative. Thus $f \circ h^{-1}$ only has critical points of type C_i in $\{0\} \times \mathbb{R}^{k-1}$, and we can apply the inductive hypothesis to $(f \circ h^{-1})|_{\{0\} \times \mathbb{R}^{k-1}}$ to see its critical values have measure 0. An argument as in the first step finishes the argument.

The case C_i**.** Finally, one proves that C_N has measure 0 for $N > k/k' - 1$. Then $C_i = (C_i \backslash C_{i+1}) \cup \cdots \cup (C_{N-1} \backslash C_N) \cup C_N$, all of which have measure 0. To see this final case, it is convenient to assume $U = (0, 1)^k$, with f extending to an open neighborhood of $[0, 1]^k$. We may make this assumption because countably many rescaled versions of closed cubes with these properties cover U. If $c \in C_N$, the Taylor approximation to order $\leqslant N$ of f at c vanishes, in the sense that $||f(c+h) - f(c)|| \leqslant D||h||^{N+1}$ for some constant $D > 0$ and $||h|| < \epsilon_0$, cf. [DK04a, Theorem 2.8.3].

Since C_N is closed in $[0,1]^k$ it is compact, and the constants D and ϵ_0 depend continuously on $c \in C_N$ we may find constants $D > 0$ and $\epsilon_0 > 0$ that work for all $c \in C_N$. Then subdivide $[0,1]^k$ into cubes with sides $1/L$ where $1/L < \epsilon_0/2$. Then f must map of each the cubes that intersects C_N into a disk of radius $\leqslant D(\sqrt{k}/L)^{N+1}$. Hence C_N is contained in a set of volume $\leqslant L^k D'(\sqrt{k}/L)^{k'(N+1)}$. If $N > k/k' - 1$ the exponent L is $< k - k'k/k' = 0$, so goes this volume goes to 0 as $L \to \infty$. □

12.5 Problems

Problem 30 (Concatenation of homotopies)**.**

(a) Suppose that $H\colon M \times [0,1] \to N$ is a homotopy from f_0 to f_1. Construct a different homotopy $\tilde{H}\colon M \times [0,1] \to N$ from f_0 to f_1 such that $\tilde{H}(-,t) = f_0$ for $t < 1/4$ and $\tilde{H}(-,t) = f_1$ for $t > 3/4$. (Hint: use bump functions.)

(b) Use part (a) to show that the relation of homotopy is transitive, i.e. $f_0 \sim f_1$ and $f_1 \sim f_2$ implies $f_0 \sim f_2$.

Problem 31 (The fundamental group)**.** For a smooth manifold M with chosen basepoint $m_0 \in M$, we consider the set of smooth maps $\gamma\colon S^1 \to M$ sending $1 \in S^1$ to $m_0 \in M$. We say that two such smooth maps $\gamma_0, \gamma_1\colon S^1 \to M$ are *homotopic relative endpoints* if there is a homotopy $H\colon M \times [0,1] \to N$ from γ_0 to γ_1 such that $H(1,t) = m_0$ for all $t \in [0,1]$.

(a) Prove that being homotopic relative to endpoints is an equivalence relation.

We denote the set of homotopy classes relative endpoints by $\pi_1(M, m_0)$, the *fundamental group M at m_0*. As the name suggests it has a group structure, which you will construct below:

(b) Use the ideas of Problem 30 to prove that concatenation of loops gives a well-defined map

$$\pi_1(M, m_0) \times \pi_1(M, m_0) \longrightarrow \pi_1(M, m_0).$$

(c) Show that concatenation makes $\pi_1(M, m_0)$ into a group. (Hint: the inverse is given by reversing loops.)

Chapter 13

Two applications of Sard's theorem

In the previous lecture we proved Sard's theorem: the set of critical values of a smooth map $f\colon M \to N$ has measure 0. Today we give two applications: (i) the strong Whitney embedding theorem, (ii) the Brouwer fixed point theorem. This is in Sections 1.§8, 2.§1 and 2.§2 of [GP10], and uses results from Appendices 1 and 2 of [GP10].

13.1 The strong Whitney embedding theorem

Let's recall the weak Whitney embedding theorem: any compact manifold M can be embedded into some Euclidean space. Today we prove the stronger statement that any compact k-dimensional manifold M can be embedded into $\mathbb{R}^{2k+1}$, and deduce from it that a non-compact k-dimensional manifold M can be embedded into $\mathbb{R}^{2k+2}$.

13.1.1 The compact case

Theorem 13.1.1 (Strong Whitney embedding theorem)**.** *If M is a compact k-dimensional smooth manifold, then there exists an embedding of M into $\mathbb{R}^{2k+1}$.*

This is a direct consequence of the following proposition using the weak Whitney embedding theorem and the fact that all injective immersions with compact domain are embeddings, since every continuous map with compact domain is proper.

Proposition 13.1.2. *If M is a k-dimensional smooth manifold with an injective immersion of M into $\mathbb{R}^N$ for some N, then there exists an injective immersion of M into $\mathbb{R}^{2k+1}$.*

Proof. If $N \leqslant 2k+1$ there is nothing to prove. If $N > 2k+1$, we will show that we can reduce N to $N-1$. Let $i\colon M \to \mathbb{R}^N$ denote the injective immersion and consider the following two smooth maps

$$\begin{aligned} f^{\mathrm{inj}}\colon M\times M\backslash\{(m,m)\mid m\in M\} &\longrightarrow S^{N-1} \\ (p,p') &\longmapsto \frac{i(p)-i(p')}{||i(p)-i(p')||}, \\ f^{\mathrm{tang}}\colon TM\backslash 0\text{-section} &\longrightarrow S^{N-1} \\ v &\longmapsto \frac{di(v)}{||di(v)||}. \end{aligned}$$

These maps were chosen because of the meaning we can ascribe to their regular values. For each $x \in S^{N-1}$ there is a linear projection $\pi_x\colon \mathbb{R}^N \to x^\perp$. If $x \notin \mathrm{im}(f^{\mathrm{inj}})$ then $\pi_x \circ i$ is injective, and if $x \notin \mathrm{im}(f^{\mathrm{tang}})$ then the derivative of $\pi_x \circ i$ is injective. In particular, if $x \notin \mathrm{im}(f^{\mathrm{inj}}) \cup \mathrm{im}(f^{\mathrm{tang}})$, then $\pi_x \circ i\colon M \to x^\perp \cong \mathbb{R}^{N-1}$ is an injective immersion of M into a Euclidean space of lower dimension.

Both $M \times M\backslash\{(m,m) \mid m \in M\}$ and $TM\backslash M$ are $2k$-dimensional. As $N-1 > 2k$, this means x is disjoint from the images of f^{inj} and f^{tang} if and only if x is a regular value of f^{inj} and f^{tang}. By Sard's theorem such joint regular values are dense so must exist. □

In fact, since the derivative is linear, to see that $\pi_x \circ i$ has injective differential, we only need to avoid the image of

$$\begin{aligned} \bar{f}^{\mathrm{tang}}\colon \{v \in TM \mid ||di(v)|| = 1\} &\longrightarrow S^{N-1} \\ v &\longmapsto di(v). \end{aligned}$$

Its domain is $(2k-1)$-dimensional, so we can go one dimension further if we only care about guaranteeing that the derivative remains injective. We can do a bit better by picking x to be a regular value of $f^{\mathrm{inj}}\colon M \times M\backslash\{(m,m) \mid m \in M\} \to S^{2k}$. In that case the intersection points of the immersion will be transverse. If M is compact, then there must be a finite number of them since transverse intersection points are isolated.

Corollary 13.1.3. *If M is a compact k-dimensional smooth manifold, then there exists an immersion of M into $\mathbb{R}^{2k}$ with finitely many transverse intersections.*

Example 13.1.4 (Whitney double point). We can always add more self-intersections, by inserting in a local chart one of the following maps, due to Whitney [Whi44, Section 1.2]. These are immersions with a single transverse double point that are approximately linear outside a compact set:

$$\begin{aligned} \alpha_k\colon \mathbb{R}^k &\longrightarrow \mathbb{R}^{2k} \\ (x_1,\ldots,x_k) &\longmapsto \left(\frac{1}{u}, x_1 - 2\frac{x_1}{u}, \frac{x_1x_2}{u}, x_2, \frac{x_1x_3}{u}, x_3, \cdots, \frac{x_1x_k}{u}, x_k\right) \end{aligned}$$

with $u = (1+x_1^2)\cdots(1+x_k^2)$. Their existence is used in the proof that every compact k-dimensional smooth manifold embeds into $\mathbb{R}^{2k}$ [Whi44, Theorem 5]. This is the best possible bound: $\mathbb{R}P^{2^n}$ does not embed in $\mathbb{R}^{2^{n+1}-1}$.

13.1.2 Non-compact case

We continue with a discussion of the non-compact case. It is based on a double application of Proposition 13.1.2 and the following lemma:

Lemma 13.1.5. *Every smooth manifold M admits a proper smooth function $\lambda\colon M \to [0,\infty)$.*

Proof. Using our lemma about compact exhaustions, we can pick compact subsets K_i and open subsets $V_{i+1/2}$ of M such that $K_0 \subset V_{1/2} \subset K_1 \subset V_{1+1/2} \subset \cdots$ and $\bigcup_i K_i = M$. Applying our result about the existence of partitions of unity, let $\eta_i\colon M \to [0,1]$ be a partition of unity subordinate to the open cover by $V_{i+1/2} \backslash K_{i-1}$. Then we define

$$\begin{aligned}\lambda\colon M &\longrightarrow [0,\infty) \\ p &\longmapsto \sum_i i\eta_i(p).\end{aligned}$$

This sum is locally finite so smooth, and if $\lambda(p) \leqslant i$ then at least one of the η_j for $j \leqslant i$ has to be non-zero, so $p \in K_{i+1}$. Thus $\lambda^{-1}([0,i])$ is a closed subset of the compact set K_{i+1} and hence λ is proper. □

Theorem 13.1.6. *If M is a k-dimensional smooth manifold, then there exists an embedding of M into some Euclidean space $\mathbb{R}^N$.*

Proof. Using once more our lemma about compact exhaustions, pick compact subsets K_i and open subsets $V_{i+1/2}$ of M such that $K_0 \subset V_{1/2} \subset K_1 \subset V_{1+1/2} \subset \cdots$ and $\bigcup_i K_i = M$. Then $K_{i+1} \backslash V_{i-1/2}$ is compact, and hence can be covered by finitely many charts. The proof of the weak Whitney embedding theorem then provides an injective immersion of an open neighbourhood W_i of $K_{i+1} \backslash V_{i-1/2}$ in $V_{i+3/2} \backslash K_{i-1}$ into some Euclidean space. By Proposition 13.1.2 we may assume this Euclidean space is in fact $\mathbb{R}^{2k+1}$.

Thus we have an open cover by $W_i \subset M$ so that $W_i \cap W_j \neq \varnothing$ is only possible if $|i-j| \leqslant 2$, which come with injective immersion $\rho_i\colon W_i \to \mathbb{R}^{2k+1}$. Now pick a partition of unity $\eta_i\colon M \to [0,1]$ subordinate to the W_i's and define smooth maps

$$p \longmapsto \overline{\eta_i(p)\rho_i(p)} := \begin{cases} \eta_i(p)\rho_i(p) & \text{if } p \in W_i, \\ 0 & \text{otherwise.}\end{cases}$$

We can then define for each i a new smooth map

$$\begin{aligned}\tilde{\rho}_i\colon M &\longrightarrow \mathbb{R}^{9(2k+2)} \\ p &\longmapsto (\eta_i(p), \overline{\eta_i(p)\rho_i(p)}) \quad \begin{array}{l}\text{put in the } j\text{th copy of } \mathbb{R}^{2k+2}, \\ 1 \leqslant j \leqslant 9, \text{ if } i \equiv j \\ \pmod 9\end{array}\end{aligned}$$

and zeroes in all other entries, and take

$$\begin{aligned}\rho\colon M &\longrightarrow \mathbb{R}^{1+9(2k+2)} \\ p &\longmapsto \left(\sum_i i\eta_i(p), \sum_i \tilde{\rho}_i(p)\right).\end{aligned}$$

This is smooth since each sum is locally finite.

This is proper because $\sum_i i\eta_i(p)$ is proper, as in Lemma 13.1.5. For each p there is an open neighborhood on which only five terms in each sum are possibly non-zero; if $p \in W_i$ then only the terms $i-2, i-1, i, i+1, i+2$ can be non-zero. In the second entry all of these open subsets map to a different copy of $\mathbb{R}^{2k+2}$, so the differential is injective by the same argument as used in the weak Whitney embedding theorem.

For injectivity, we further observe that if $p \in W_i$ then $i-2 \leqslant \sum_i i\eta_i(p) \leqslant i+2$. That is, if $l := \sum_i i\eta_i(p)$, then $p \in \bigcup_{j=-2}^{2} W_{\lceil l \rceil + j}$. From this we conclude that if $\rho(p) = \rho(p')$, then both p and p' are in $\bigcup_{j=-2}^{2} W_{\lceil l \rceil + j}$. On this open subset only nine terms in the second sum are possibly non-zero, all of which map to a different copy of $\mathbb{R}^{2k+2}$. Again we can apply the proof of the weak Whitney embedding theorem to deduce injectivity. □

In fact, we can now reduce the dimension again:

Corollary 13.1.7. *If M is a k-dimensional smooth manifold, then there exists an embedding of M into $\mathbb{R}^{2k+2}$.*

Proof. We start with an embedding as in the previous lemma. Proposition 13.1.2 gives us an injective immersion of M into $\mathbb{R}^{2k+1}$. If we pick a proper smooth function $\lambda\colon M \to [0,\infty)$ as in Lemma 13.1.5, we get an embedding $i := (\lambda, e)\colon M \to \mathbb{R}^{2k+2}$. □

Remark 13.1.8. In fact, by the argument on pp. 53–54 of [GP10] you can decrease the dimension once more to get an embedding $M \hookrightarrow \mathbb{R}^{2k+1}$ by a projecting along a suitable $x \in S^{2k+1}$.

13.2 Manifolds with boundary

A k-dimensional smooth manifold M is a second countable Hausdorff space with a k-dimensional smooth atlas. The atlas provides a local identification of M with an open subset of $\mathbb{R}^k$, such that transition functions are smooth.

Unfortunately, using these definitions such reasonable spaces as D^n and $M \times [0,1]$ are *not* smooth manifolds, because a point in ∂D^n resp. $M \times \{0,1\}$ does not admit an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^k$. To allow these examples, we need to broaden our scope and consider manifolds to have boundary. These are locally modelled on $[0,\infty) \times \mathbb{R}^{k-1}$ instead of $\mathbb{R}^k$.

13.2.1 Definitions

Definition 13.2.1. A *k-dimensional smooth atlas with boundary* for topological space M is a collection of triples $(U_\alpha, V_\alpha, \phi_\alpha)$ consisting of open subsets $U_\alpha \subset [0,\infty) \times \mathbb{R}^{k-1}$, $V_\alpha \subset M$ and homeomorphisms $\phi_\alpha\colon U_\alpha \to V_\alpha$, so that $\bigcup V_\alpha = X$ and all maps

$$\phi_\beta^{-1} \circ \phi_\alpha\colon \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \longrightarrow \phi_\beta^{-1}(V_\alpha \cap V_\beta)$$

are smooth maps between open subsets of $[0,\infty) \times \mathbb{R}^{k-1}$ (they are then automatically diffeomorphisms since they have smooth inverses). The triples $(U_\alpha, V_\alpha, \phi_\alpha)$ are called *charts* and the maps $\phi_\beta^{-1} \circ \phi_\alpha$ are called *transition functions.*

Here we use that we already know what a smooth map between open subsets of $[0,\infty) \times \mathbb{R}^{k-1}$ is; it is a function which locally extends to a smooth function on an open subset of $\mathbb{R}^k$. All of the previously discussed machinery goes through, starting with the definitions:

Lemma 13.2.2. *Every k-dimensional smooth atlas with boundary is contained in a unique maximal k-dimensional smooth atlas with boundary.*

Definition 13.2.3. A *k-dimensional smooth manifold with boundary* is a Hausdorff second countable topological space X with a maximal k-dimensional smooth atlas with boundary.

Example 13.2.4. If M is a k-dimensional smooth manifold in the ordinary sense, it is also a k-dimensional smooth manifold with boundary. Its boundary just happens to be empty.

Example 13.2.5. If M is a $(k-1)$-dimensional smooth manifold, then $M \times [0,1]$ is a k-dimensional smooth manifold with boundary.

Suppose that a diffeomorphism between open subsets of $[0,\infty) \times \mathbb{R}^{k-1}$ sends a point in $(0,\infty) \times \mathbb{R}^{k-1}$ to a point in $\{0\} \times \mathbb{R}^{k-1}$. Its derivative is bijective, so the inverse function says it is local diffeomorphism. This means that it must also hit some points in $(-\infty,0) \times \mathbb{R}^{k-1}$, which is not allowed. Hence a diffeomorphism must send points in $\{0\} \times \mathbb{R}^{k-1}$ to points in $\{0\} \times \mathbb{R}^{k-1}$. Hence the following is a reasonable definition:

Definition 13.2.6. The *boundary ∂M* of a k-dimensional smooth manifold M with boundary is the subset of those points that are in the image of $\{0\} \times \mathbb{R}^{k-1}$ under a chart.

The charts of M restrict to charts for ∂M, and we get a smooth $(k-1)$-dimensional atlas for ∂M.

13.2.2 Theorems

Let us now explain the modifications that need to be made to the theory when including manifolds with boundary. We will only state the results here, you should read their proofs in 2.§1 of [GP10].

We can give the definitions of a smooth map between manifolds with boundary, tangent bundles, and derivatives, as before. These behave with respect to the boundary as follows: a smooth map $f\colon M \to N$ between manifolds with boundary restricts to a smooth map $\partial f\colon \partial M \to N$. At $p \in \partial M$, the tangent space $T_p\partial M$ is a $(k-1)$-dimensional linear subspace of T_pM, and $d_p\partial f = (d_pf)|_{T_p\partial M}$.

The pre-image theorem and Sard's lemma generalize in the following manner:

Theorem 13.2.7 (Pre-image theorem for manifolds with boundary)**.** *Let $f: M \to N$ be a smooth map with M a manifold with boundary, N a manifold without boundary, and $Z \subset N$ a submanifold without boundary. If $f \pitchfork Z$ and $\partial f \pitchfork Z$, then $f^{-1}(Z) \subset M$ is a manifold with boundary $\partial(f^{-1}(Z)) = (\partial f)^{-1}(Z)$. Moreover, the codimension of $f^{-1}(Z)$ is equal to the codimension of Z and $Tf^{-1}(Z) = df^{-1}(TZ)$.*

Theorem 13.2.8 (Sard's theorem for manifolds with boundary)**.** *For any smooth map $f: M \to N$ with M a manifold with boundary and N a manifold without boundary, the subset of points in N which are critical values of f or ∂f has measure* 0*.*

13.3 The Brouwer fixed point theorem

The Brouwer fixed point theorem says that every continuous map $F: D^n \to D^n$ has a fixed point. This is deduced from the theorem that there are no continuous maps $f: D^n \to \partial D^n$ which are the identity on ∂D^n.

We will prove a version of this result, which is stronger because it concerns all manifolds with boundary, but weaker because it concerns only smooth maps. The latter is however easily remedied by the use of certain smooth approximation results. To prove our generalisation we use another fact, which is proven in Appendix 2 of [GP10] or the Appendix of [Mil97].

Theorem 13.3.1 (Classification of 1-dimensional manifolds)**.** *Every compact connected* 1*-dimensional manifold is diffeomorphic to either S^1 or* $[0,1]$.

Corollary 13.3.2. *The boundary of every compact* 1*-dimensional manifold is an even number of points.*

Using this trivial observation, we prove Hirsch's generalization of the Brouwer fixed point theorem:

Theorem 13.3.3 (Hirsch)**.** *Let M be a compact manifold with boundary. Then there is no smooth map $M \to \partial M$ which is the identity on ∂M.*

Proof. Suppose for the sake of contradiction that such an $f: M \to \partial M$ does exist. By Sard's theorem we can pick an $p \in \partial M$ which is a regular value of both f and ∂f. This means that $f^{-1}(p) \subset M$ is a 1-dimensional manifold with boundary. It is closed in a compact space hence compact, and thus by Theorem 13.3.1 has an even number of boundary points. But $\partial f^{-1}(p) = (\partial f)^{-1}(p) = \{p\}$ since $\partial f = \mathrm{id}_{\partial M}$. This is a contradiction. □

Remark 13.3.4. One easily generalizes this proof to say that there is no smooth map $M \to \partial M$ which is injective on ∂M.

Let us deduce from this the Brouwer fixed point theorem for smooth maps:

Corollary 13.3.5 (Smooth Brouwer fixed point theorem)**.** *If $F: D^n \to D^n$ is a smooth map, it has a fixed point.*

Proof. For a proof by contradiction, we suppose that F has no fixed points. Then

$$\begin{aligned} f\colon D^n &\longrightarrow \partial D^n \\ x &\longmapsto \text{intersection with } \partial D^n \text{ of half-line starting at } F(x) \text{ through } x \end{aligned}$$

is a well-defined smooth function $f\colon D^n \to \partial D^n$ that is the identity on ∂D^n. □

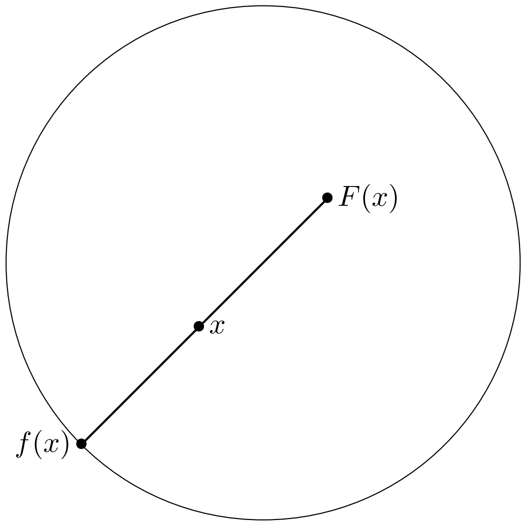


Figure 13.1 The map f in the proof of the Brouwer fixed point theorem.

Example 13.3.6. There is an anecdotal application of the Brouwer fixed point theorem to physics. Trying to balance a pencil on a table, it seems intuitive that there is an equilibrium point. You can of course prove this in an idealised setting, but it seems hard if we use some realistic model of the forces acting upon and within the pencil.

Suppose there is no equilibrium point, then the pencil would always fall with eraser facing some direction. This gives a map from the upper hemisphere S^2_+ to S^1, which is clearly the identity on the boundary. The claim is that the Brouwer fixed point theorem rules this out, so an equilibrium point must exist. However, it is far from obvious that the described map is continuous (see the section "Courant–Robbins Train" of [Ste11]).

13.4 Problems

Problem 32 (Classification of 1-dimensional manifolds)**.** Read Appendix 2 of [GP10] and the Appendix of [Mil56b]. Which proof of the classification of 1-dimensional smooth manifolds do you prefer, and why?

Chapter 14

Transverse maps are generic

Today we prove a result announced in earlier: the set of smooth maps $f\colon M \to N$ transverse to $Z \subset M$ is generic. As an application we deduce the tubular neighbourhood theorem. This is 2.§3 of [GP10].

14.1 Transverse maps are generic

Recall the following definition:

Definition 14.1.1. A subset D of the set of all smooth maps $M \to N$ is *generic* if for all $f_0\colon M \to N$ we can find a smooth map $H\colon M \times \mathbb{R}^r \to N$ with $H|_{M\times\{0\}} = f_0$ so that for all $\epsilon > 0$ there exists an $x \in \mathbb{R}^r$ with $||x|| < \epsilon$ such that $H|_{M\times\{x\}} \in D$.

The "perturbation" $H\colon M \times \mathbb{R}^r \to N$ is a particular example of a family of smooth maps as below, where $S = \mathbb{R}^r$. This is just a change of perspective; we think of F not as a single map $M \times S \to N$ but a collection of maps $M \to N$ parametrized by S.

Definition 14.1.2. Let S be a smooth manifold, then a *family of smooth maps $M \to N$ indexed by S* is a smooth map $F\colon M \times S \to N$.

Since the restriction of a smooth map to a submanifold is smooth, $f_s := F|_{M\times\{s\}}$ is a smooth map for each $s \in S$.

Theorem 14.1.3. *Suppose that $F\colon M \times S \to N$ is a family of smooth maps $M \to N$, where M may have boundary but S and N do not. Let $Z \subset N$ be a submanifold without boundary. If $F \pitchfork Z$ and $\partial F \pitchfork Z$, then there is a dense set of $s \in S$ such that $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$.*

As usual when applying Sard's theorem, we will actually prove that the complement of those $s \in S$ such that $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ has measure zero.

Proof. Let $W = f^{-1}(Z) \subset M \times S$, a submanifold with boundary $\partial W = W \cap (\partial M \times S)$ by the improved preimage theorem. Thus we can ask for regular values of the restriction $\pi|_W\colon W \to S$ of the projection $M \times S \to S$, as well as its restriction $\pi|_{\partial W}\colon \partial W \to S$ to the boundary. Such common regular values are dense by Sard's theorem.

We claim that $f_s \pitchfork Z$ if and only if s is a regular value of $\pi|_W$, and similarly $\partial f_s \pitchfork Z$ if and only if s is a regular value of $\pi|_{\partial W}$. Let us only prove the first equivalence, the second one being similar.

Let us first use the hypothesis that $F \pitchfork Z$, at $(p,s) \in W \subset M \times S$ mapping to $z \in Z$ under F. Then the projections induce a linear isomorphism $T_{p,s}(M \times S) \cong T_pM \oplus T_sS$, and transversality exactly means that

$$d_{(p,s)}F(T_pM \oplus T_sS) + T_zZ = T_zN.$$

By the preimage theorem, we may describe $T_{(p,s)}W$ as $(d_{(p,s)}F)^{-1}(T_zZ) \subset T_pM \oplus T_sS$. The derivative $d_{(p,s)}\pi|_W\colon T_{(p,s)}W \to T_sS$ is the restriction of projection $T_pM \oplus T_sS \to T_sS$ to this subspace.

We next want to show that $d_{(p,s)}\pi|_W\colon T_{(p,s)}W \to T_sS$ is surjective if and only if $d_{(p,s)}F(T_pM) + T_zZ = T_zN$. This is the linear-algebraic lemma following this proof, applied with $U = T_pM$, $U' = T_sS$, $V = T_zN$, $W = T_zZ$, $T = d_{(p,s)}F$.

Finally, observe that because $d_{(p,s)}F(T_pM) = d_pf_s(T_pM)$, the statement $d_{(p,s)}F(T_pM) + T_zZ = T_zN$ is true if and only if $f_s \pitchfork Z$ at z. □

Lemma 14.1.4. *If $TA\colon U \oplus U' \to V$ is a linear map of finite-dimensional vector spaces, $W \subset V$ such that $A(U \oplus U') + W = V$. Then $\pi_2\colon A^{-1}(W) \to U'$ is surjective if and only if $A(U) + W = V$.*

Proof. For $\Rightarrow$; if $\pi_2\colon A^{-1}(W) \to U'$ is surjective, it admits a section $s\colon U' \to A^{-1}(W)$. Then we have

$$A(u+u') + W = A(u+u') + A(-s(u')) + W = A(u+u'-s(u')),$$

and since $\pi_2(u+u'-s(u')) = 0$, $u+u'-s(u') \in U$.

For $\Leftarrow$, take $u' \in U'$ and note that because $A(U) + W = V$ we can find $u \in U$ and $w \in W$ such that $A(u') = A(u) + w$. Then $u' - u \in A^{-1}(W)$ and $\pi_2(u'-u) = u'$. □

We will now prove the maps transverse to Z are generic by showing that for every $f_0\colon M \to N$ there exists a smooth map $F\colon M \times \mathbb{R}^r \to N$ such that $F|_{M\times\{0\}} = f_0$ and which satisfies $F \pitchfork Z$ and $\partial F \pitchfork Z$. To construct F we shall embed N into an Euclidean space $\mathbb{R}^r$ using the weak Whitney embedding theorem, and consider the rather uninteresting family

$$\begin{aligned} \tilde{F}\colon M \times \mathbb{R}^r &\longrightarrow \mathbb{R}^r \\ (p,s) &\longmapsto f_0(p) + s \end{aligned}$$

This is obviously a submersion so transverse to the submanifold $Z \subset N \subset \mathbb{R}^r$, and by the previous theorem there is a dense set of $s \in \mathbb{R}^r$ such that $\tilde{f}_s := \tilde{F}|_{M\times\{s\}}$ is transverse to Z. The problem is now that $\tilde{f}_s$ does not map M into N any more. To fix this, we shall use the following theorem to "project back into N":

Theorem 14.1.5 (Regular neighbourhood theorem)**.** *For every submanifold $N \hookrightarrow \mathbb{R}^r$ without boundary, there exists an open neighbourhood $U \subset \mathbb{R}^r$ of N with a submersion $\pi_N\colon U \to N$ that is the identity on N. At $n \in N \subset U$, the linear map $d_n\pi\colon T_n\mathbb{R}^r = T_nN \oplus T_nN^{\perp} \to T_nN$ is given by orthogonal projection onto T_nN.*

Remark 14.1.6. In fact, if M is compact U can be obtained by picking a small enough $\epsilon > 0$, letting U be the set of points of distance $< \epsilon$ to N and π_N be the map sending $x \in U$ to the unique closest point in N (so implicitly we are saying you can find an $\epsilon > 0$ such that this exists and is unique). This follows from the proof of Theorem 14.2.4. For non-compact M, ϵ is replaced by a smooth positive-valued function.

We shall prove the regular neighbourhood theorem in Section 14.2, and first finish the proof of genericity.

Theorem 14.1.7. *Suppose M is a manifold possibly with boundary, N is a manifold without boundary and $Z \subset N$ is a submanifold without boundary. If $f_0\colon M \to N$ is a smooth map, then there exists an $r \geqslant 0$ and a smooth map $H\colon M \times \mathbb{R}^r \to N$ starting at f_0 so that for all $\epsilon > 0$ there exists an $x \in \mathbb{R}^r$ with $||x|| < \epsilon$ such that $H|_{M\times\{x\}}$ is transverse to Z.*

Proof. Embed N into a Euclidean space $\mathbb{R}^r$ and identify N with its image in $\mathbb{R}^r$. Take $U \subset \mathbb{R}^r$ and $\pi_N\colon U \to N$ as in the regular neighbourhood theorem. Since $U \subset N$ is an open neighbourhood, we can find a smooth function $\epsilon\colon N \to (0,\infty)$ such that for each $p' \in N$ and $x \in \mathbb{R}^r$ satisfying $||x|| < \epsilon(p')$, $p' + x \in U$, see **??**. Then we define the smooth map

$$\begin{aligned} F\colon M \times \mathbb{R}^r &\longrightarrow N \\ (p,s) &\longmapsto \pi_N\left(f_0(p) + \epsilon(f_0(p))\frac{s}{1+||s||^2}\right). \end{aligned}$$

By construction, $F|_{M\times\{0\}} = \pi_N \circ f_0 = f_0$ because π_N is the identity on N.

Since π_N is a submersion, F is a submersion if and only if the map $M\times\mathbb{R}^r \to U$ given by $(p,s) \mapsto f_0(p) + \epsilon(f_0(p))\frac{s}{1+||s||^2}$ is. But when we fix $p \in M$ this is a diffeomorphism of $\mathbb{R}^r$ onto a little ball, so has surjective differential at each point in $M \times \mathbb{R}^r$. The same argument shows that $\partial F\colon \partial M \times \mathbb{R}^r \to N$ is a submersion.

Now that we have established that F and ∂F are submersions, they are clearly transverse to Z and Theorem 14.1.3 gives the desired conclusion. □

Picking a point $s \in \mathbb{R}^r$ such that $F|_{M\times\{s\}} \pitchfork Z$ and $\partial F|_{\partial M\times\{s\}} \pitchfork Z$, the homotopy $H\colon M \times [0,1] \to N$ given by $(p,t) \mapsto F(p,ts)$ proves:

Corollary 14.1.8. *Suppose M is a manifold possibly with boundary, N is a manifold without boundary and $Z \subset N$ is a submanifold without boundary. Then any smooth map $f_0\colon M \to N$ is homotopic to $f_1\colon M \to N$ satisfying $f_1 \pitchfork Z$ and $\partial f_1 \pitchfork Z$.*

14.1.1 Isotoping submanifolds

In the introduction, we discussed how to deform embeddings. This is the notion of isotopy, intuitively a one-parameter family of embeddings. Let us recall the definition:

Definition 14.1.9. A homotopy $H\colon M \times [0,1] \to N$ is an *isotopy* if $M \times [0,1] \ni (x,t) \mapsto (H(m,t),t) \in N \times [0,1]$ is an embedding.

This is implied by H being a smooth proper map such that $H|_{M\times\{t\}}$ is an embedding for all $t \in [0,1]$, as then the map $M \times [0,1] \to N \times [0,1]$ is a proper injective immersion. Note that if M is compact, we may drop the hypothesis that this map is proper.

Suppose we are given two submanifolds $Y, Z \subset M$ without boundary, with Y compact. We can then consider the inclusion $i\colon Y \hookrightarrow M$ as a smooth map. By Theorem 14.1.7 we can find a map $F\colon Y \times \mathbb{R}^r \to M$ such that the set of $s \in \mathbb{R}^r$ such that $F|_{Y\times\{s\}}\colon Y \to M$ is transverse to Z, is dense.

Since the class of embeddings is stable when the domain is compact, we can find $\epsilon > 0$ such that $F|_{Y\times\{s\}}\colon Y \to M$ is still an embedding if $||s|| < \epsilon$. Take such an s with $H|_{Y\times\{s\}} \pitchfork Z$. Then the homotopy

$$\begin{aligned} H\colon Y \times [0,1] &\longrightarrow M \\ (y,t) &\longmapsto F(y,ts) \end{aligned}$$

is an isotopy of embeddings of Y into M from i to an embedding transverse to Z, a strengthening of Corollary 14.1.8.

Informally, the maps $H|_{Y\times\{t\}}$ tells us how to move submanifold Y to a new position at which it is transverse to Z. If Y is r-dimensional and Z is s-dimensional, satisfying $r + s < k$, then Y is transverse to Z if and only if $Y \cap Z = \varnothing$. Thus we have shown that in these conditions any two submanifolds can be made disjoint by moving one of them.

Example 14.1.10. Suppose we take $S^1 = \{(x,y,0) \mid x^2 + y^2\} \subset \mathbb{R}^3$ and any other embedding $i\colon S^1 \to \mathbb{R}^3$. This gives us two submanifolds of $\mathbb{R}^3$ which are diffeomorphic to S^1. They may very well be linked in a complicated way in $\mathbb{R}^3$. However, if we increase the dimension by 1 they become unlinked. That is, we claim that we can isotope $i(S^1) \subset \mathbb{R}^4$ in the complement of $S^1 \subset \mathbb{R}^4$ so that it becomes disjoint from the disk $D^2 \subset \mathbb{R}^4$. This follows by applying the above observations with $Y = i(S^1)$, $Z = D^2 \backslash S^1$ and $N = \mathbb{R}^4 \backslash S^1$, as the dimensions of Y and Z add up to $3 < 4$.

14.2 The regular neighbourhood theorem

It remains to prove Theorem 14.1.5. This uses a new vector bundle associated to a submanifold $Z \subset M$, the *normal bundle*. Over Z we have two vector bundles, the trivial bundle $TM|_Z$ and its subbundle TZ.

Definition 14.2.1. The *normal bundle* NZ is the vector bundle over Z given by $TM|_Z/TZ$.

When $M = \mathbb{R}^r$ and $Z = N$, this admits a more concrete definition. In that case $T\mathbb{R}^r|_N = N \times \mathbb{R}^r$ and comes with a preferred inner product on each fibre (the restriction of usual Euclidean inner product). The orthogonal complements $(T_pN)^\perp$ assemble to a vector bundle $TN^\perp$ over N, explicitly given by

$$\{(p, v) \in N \times \mathbb{R}^r \mid v \perp T_pN\}.$$

Orthogonal projection gives a map $T\mathbb{R}^r|_N \to TN^\perp$ whose kernel is exactly TN. Thus there is an induced isomorphism

$$T\mathbb{R}^r|_N/TN \xrightarrow{\cong} TN^\perp$$

of vector bundles over N.

Example 14.2.2. Let us verify $TN^\perp$ is a vector bundle. Suppose we have a local trivialization $\phi\colon N \cap V \cong U \cap \mathbb{R}^{k'} \subset \mathbb{R}^r$. For $x \in \mathbb{R}^{k'}$, the bilinear map $(v, v') \mapsto \langle d_x\phi(v), d_x\phi(v')\rangle$ is an inner product on $T_x\mathbb{R}^r$. We can think of this as a symmetric matrix A_x whose entries vary smoothly with x as follows: $v \cdot A_xv' = \langle d_x\phi(v), d_x\phi(v')\rangle$. Every positive semidefinite symmetric matrix A has a unique decomposition $A = B^tB$ with B again positive semidefinite, and the entries of B depend smoothly on those of A. Thus we can identify $TN^\perp$ with the subbundle

$$\{(x, B^{-1}v) \mid x \in \mathbb{R}^{k'},\ v \in \{0\} \times \mathbb{R}^{k'-r}\} \subset (\mathbb{R}^{k'} \times \mathbb{R}^r),$$

visibly admitting a local trivialization.

Furthermore, it is clear from this description that the transitions between local trivializations are smooth, so $TN^\perp$ is a smooth vector bundle. In particular, $TN^\perp$ is a manifold and the projection map $\pi\colon TN^\perp \to N$ is a submersion.

We now prove the regular neighbourhood theorem, which said that given a $N \hookrightarrow \mathbb{R}^r$ without boundary, there exists an open neighbourhood $U \subset \mathbb{R}^r$ of N and a submersion $\pi_N\colon U \to N$ that is the identity on N. Furthermore, the linear map $d_p\pi\colon T_p\mathbb{R}^r = T_pN \oplus T_pN^\perp \to T_pN$ is given by orthogonal projection onto T_pN.

Proof of Theorem 14.1.5. Define the smooth map

$$\begin{aligned} h\colon TN^\perp &\longrightarrow \mathbb{R}^r \\ (p, v) &\longmapsto p + v. \end{aligned}$$

Because $TN^\perp$ is r-dimensional, so is the tangent space $T_{(p,0)}N^\perp$. As the manifold $TN^\perp$ contains the submanifolds $N \times \{0\}$ and $\{p\} \times T_pN^\perp$, which intersect only at $(p, 0)$, $T_{(p,0)}N^\perp$ contains their tangent spaces at $(p, 0)$, given by T_pN and $T_pN^\perp$ respectively. This gives a linear map

$$T_pN \oplus T_pN^\perp \longrightarrow T_{(p,0)}TN^\perp,$$

which we claim is an isomorphism. Since both sides have the same dimension, and this map is an inclusion on each summand, it suffices to prove that T_pN and $T_pN^\perp$

intersect only in $\{0\}$. This follows from the fact that the map $d_{(p,0)}\pi\colon T_{(p,0)}N^\perp \to T_pN$ is the identity on T_pN and 0 on $T_pN^\perp$

With respect to this direct sum decomposition, the linear map

$$d_{(p,0)}h\colon T_{(p,0)}TN^\perp \longrightarrow \mathbb{R}^r$$

is given by sending the summand T_pN onto $T_pN \subset T_p\mathbb{R}^r$ and the summand $T_pN^\perp$ to $T_pN^\perp \subset T_p\mathbb{R}^r$. In particular, it is bijective.

By the inverse function theorem, it is a local diffeomorphism near N. As it is an embedding on N, it is injective on an open neighbourhood V of N by Lemma 14.2.3 (take $A = N$, $M = TN^\perp$, $N = \mathbb{R}^r$) and hence gives a diffeomorphism $h\colon V \to U$ for $U := h(V)$ an open neighbourhood of N in $\mathbb{R}^r$. Now set

$$\pi_N := \pi \circ h^{-1}\colon U \longrightarrow V \longrightarrow N.$$

Since π_N is a composition of a diffeomorphism and a submersion, it is a submersion. Since π and h are the identity on N, so is π_N. To prove the addendum, it remains to observe that $d_{(p,0)}\pi\colon T_{(p,0)}TN^\perp \cong T_pN \oplus T_pN^\perp \to T_pN$ is projection onto the first summand. □

Lemma 14.2.3. *If $A \subset M$ is closed and $f\colon M \to N$ is a smooth map which is a local diffeomorphism near A and injective on A, then f is injective near A.*

Proof. We first this prove in the case that A is compact. For contradiction, suppose there is pair of sequence of points $p_i \in M$, $p_i' \in M$ so that $p_i \neq p_i'$, $f(p_i) = f(p_i')$, which get arbitrarily close to A. By compactness of A, we may assume they converge: $p_i \to p \in A$ and $p_i' \to p' \in A$. Then by continuity $f(p) = f(p')$, so $p = p'$ since f is injective on p. But since f is a local diffeomorphism near p it is injective near p and hence $p_i = p_i'$ for i large enough.

In the general case, take the subset $D = \{(p, p') \in M \times M \mid p \neq p',\ f(p) = f(p')\}$. By assumption on A, it is disjoint from $A \times A$. Its closure is contained in the union of D with the diagonal, but the local diffeomorphism condition implies that every point in the diagonal has an open neighbourhood disjoint from D. Thus D is closed and its complement is open. By exhausting M with compact subsets and applying the above argument, this open subset contains a product neighbourhood $W_{p,q} \times W'_{p,q} \subset M \times M$ of each point $(p, q) \in A \times A$; by replacing $W_{p,q}$ with $W_{p,q} \cap W'_{q,p}$ we may assume that $W_{p,q} = W_{q,p}$ for all $(p, q) \in A \times A$. Then $\bigcup_{p,q} W_{p,q} \subset M$ is the desired open neighbourhood. □

14.2.1 The tubular neighbourhood theorem

We will now slightly generalize Theorem 14.1.5, replacing $\mathbb{R}^r$ with an arbitrary manifold and choosing a smaller but nicer neighbourhood:

Theorem 14.2.4 (Tubular neighbourhood theorem)**.** *For every submanifold $Z \hookrightarrow N$, there is an open neighbourhood W of Z in N and a diffeomorphism $\phi\colon NZ \longrightarrow W$ that is the identity on Z.*

Proof. Given an embedding $N \hookrightarrow \mathbb{R}^r$, let U, V and π_N be as in the proof of the regular neighbourhood theorem. We can then identify NZ with the orthogonal complement $TZ^\perp$ of TZ in TN. The only thing we will use of this observation is that the orthogonal projection map $TN|_Z \to NZ$ has a section.

Define the smooth map

$$\begin{aligned} \tilde{h} \colon NZ &\longrightarrow \mathbb{R}^r \\ (p, v) &\longmapsto p + v. \end{aligned}$$

and take $W' = \tilde{h}^{-1}(V)$. The map $\pi_N \circ \tilde{h}$ has bijective differential at the 0-section, because by the chain rule it is the composition of $d_p\tilde{h} \colon T_pZ \oplus NZ \cong T_pN \to T_pN$, which is the identity, and $T_p\pi_N \colon T_p\mathbb{R}^r \cong T_pN \oplus T_pN^\perp \to T_pN$ the projection onto the first summand. It also is the identity on N. By the same argument as before, we find an open neighbourhood W'' of the 0-section in NZ on which $\pi_N \circ \tilde{h}$ is an embedding. We can find a smooth function $\epsilon \colon Z \to (0, \infty)$ such that

$$W = \{(p, v) \in NZ \mid ||v|| < \epsilon(p)\} \subset W'',$$

where $||-||$ is the norm from the inner product on $TZ^\perp \subset \mathbb{R}^r$. The diffeomorphism is given by

$$\begin{aligned} \phi \colon \nu_Z &\longrightarrow W \\ (p, v) &\longmapsto \left(p, \epsilon(p)\frac{v}{1 + ||v||^2}\right). \end{aligned}$$

This completes the proof. □

14.2.2 Collars

In a manifold M with boundary ∂M, the boundary admits particularly nice open neighbourhoods:

Definition 14.2.5. A *collar of* ∂M is a open neighbourhood $V \subset M$ of ∂M with a diffeomorphism $\phi \colon V \to \partial M \times [0, 1)$ that is the identity on ∂M.

Theorem 14.2.6. *Every manifold with boundary admits a collar.*

We will construct the two components $V \colon [0, 1)$ and $V \to \partial M$ independently.

Lemma 14.2.7. *There exists a smooth map* $\chi \colon M \to [0, \infty)$ *such that*

(i) $\chi^{-1}(0) = \partial M$ *and*

(ii) for each $p \in \partial M$ *there exists a* $v \in T_pM \backslash T_p\partial M$ *with* $d\chi(v) \neq 0$.

Proof. Pick charts $\phi_\alpha \colon \mathbb{R}^{k-1} \times [0, \infty) \supset U_\alpha \to V_\alpha \subset M$ whose codomains cover M. The local coordinates gives a smooth function

$$\begin{aligned} f_\alpha \colon V_\alpha &\longrightarrow [0, \infty) \\ p &\longmapsto \pi_2 \circ \phi_\alpha^{-1}(p), \end{aligned}$$

with $\pi_2\colon \mathbb{R}^{k-1} \times [0,\infty) \to [0,\infty)$ the projection onto the second coordinate.

Let us now pick a partition of unity subordinate to the open cover $\{V_\alpha\}$, given by smooth functions $\lambda_\alpha\colon V_\alpha \to [0,1]$. The function $\lambda_\alpha f_\alpha$ extends by zero to a smooth function $\overline{\lambda_\alpha f_\alpha}\colon M \to [0,\infty)$. Then the function

$$\begin{aligned} \chi\colon M &\longrightarrow [0,\infty) \\ p &\longmapsto \sum_\alpha \overline{\lambda_\alpha f_\alpha}(p) \end{aligned}$$

has the desired properties. We will leave the verification of this to the reader. □

Lemma 14.2.8. *There exists an open neighbourhood $U \subset M$ of ∂M with a smooth map $r\colon V \to \partial M$ that is the identity on ∂M.*

Proof. The weak Whitney embedding theorem also holds for manifolds with boundary, so we may pick an embedding $e\colon M \hookrightarrow \mathbb{R}^N$ and consider M as a submanifold of Euclidean space. We may then apply the regular neighbourhood theorem, Theorem 14.1.5, to ∂M, resulting in an open neighbourhood $U \subset \mathbb{R}^N$ of ∂M with a smooth map $\pi_{\partial M}\colon U \to \partial M$ that is the identity on ∂M. We then have $V := U \cap M$ and $r = \pi_{\partial M}|_V$. □

Proof of Theorem 14.2.6. We may combine $\chi|_V$ and r to a smooth map

$$f := r \times \chi|_V \colon V \longrightarrow M \times [0,\infty).$$

By construction, this is the identity and has bijective derivative on ∂M. By the inverse function theorem, it is thus a local diffeomorphism near ∂M. As a consequence of Lemma 14.2.3, it is injective onto some smaller open neighbourhood V' of ∂M. Picking a smooth function $\epsilon\colon \partial M \to (0,\infty)$ such that $\{(q,t) \in \partial M \times [0,\infty) \mid t \in [0,\epsilon(q))\} \subset f(V')$. Setting

$$\begin{aligned} U &:= f^{-1}(\{(q,t) \in \partial M \times [0,\infty) \mid t \in [0,\epsilon(q))\}) \\ \phi &:= f|_U \end{aligned}$$

is the desired diffeomorphism. □

Collars are unique up to isotopy. They have great use in reducing questions about manifolds with boundary to separate questions about the boundary and the interior.

14.3 Problems

Problem 33 (Transversality and normal bundles)**.** Let $Y, Z \subset N$ be submanifolds. Prove that $Y \pitchfork Z$ if and only if for all $p \in Y \cap Z$, $N_pY \cap N_pZ = \{0\}$.

Problem 34 (Smooth ϵ)**.** Prove that $N \subset \mathbb{R}^r$ is a submanifold and $U \subset \mathbb{R}^r$ is an open neighbourhood of N, there exists a smooth function $\epsilon\colon N \to (0,\infty)$ so that $p + x \in U$ for $p \in N$ and $x \in \mathbb{R}^r$ with $||x|| < \epsilon(p)$. (Hint: prove this exists locally in N and use a partition of unity.)

Problem 35 (Collared embeddings)**.** Use collars to prove that there exists an embedding $e\colon M \to \mathbb{R}^N \times [0,\infty)$ such that $e^{-1}(\mathbb{R}^N \times \{0\}) = \partial M$.

Problem 36 (Smooth maps and submanifolds)**.** Suppose that $X \subset M$ is a submanifold. Prove that a continuous map $f\colon X \to N$ is smooth if and only if it extends to a smooth amp $\tilde{f}\colon M \to N$.

Problem 37 (Smooth approximation)**.** It is a consequence of the *Stone-Weierstrass approximation theorem* that for all open subsets $U \subset \mathbb{R}^k$, compact subsets $K \subset U$, $\epsilon > 0$, and continuous maps $f\colon U \to \mathbb{R}$, there exists a smooth map $g\colon U \to \mathbb{R}$ such that $|g(x) - f(x)| < \epsilon$ for all $x \in K$.

(a) Prove that for each compact k-dimensional smooth manifold M, $\epsilon > 0$, and continuous map $f\colon M \to \mathbb{R}$, there exists a smooth map $g\colon M \to \mathbb{R}$ such that $|g(x) - f(x)| < \epsilon$ for all $x \in M$.

(b) Is this result still true when we drop the assumption that M is compact?

Problem 38 (Gluing manifolds with boundary)**.** Suppose that M_0 and M_1 are d-dimensional smooth manifolds, and that we are given a diffeomorphism $\varphi\colon \partial M_0 \to \partial M_1$. Use the existence of collars to produce a smooth structure on the topological space $M_0 \cup_\varphi M_1$ such that the inclusions $M_0 \to M_0 \cup_\varphi M_1$ and $M_1 \to M_0 \cup_\varphi M_1$ are smooth embeddings.

Chapter 15

Mod 2 intersection theory

In this lecture we use a slight technical strengthening of the theorem that transverse maps are generic to develop mod 2 intersection theory; this constructs invariants by counting transverse intersection points.

15.1 A strongly relative transversality theorem

Two lectures ago we proved that if M and N are manifolds without boundary, and $Z \subset M$ is a submanifold without boundary, then any smooth map $f_0 \colon M \to N$ can be homotoped to a map $f_1 \colon M \to N$ such that $f_1 \pitchfork Z$.

Sometimes you already know that f_0 is transverse to Z on an open neighbourhood U of closed subset $C \subset M$, and you do not want to modify f_0 near C. In fact, you might want to control more precisely where you modify f_0 and fix a closed subset $D \subset M$ (where we definitely want to modify f_0) and an open subset $V \subset M$ containing $D \backslash U$ (outside of which we definitely do not want to modify f_0). Many results in differential topology admit such refined forms, which are referred to as *strongly relative* results.

Theorem 15.1.1 (Strongly relative transversality theorem)**.** *Suppose that M is a compact manifold with boundary, N is a manifold without boundary, and Z is a submanifold without boundary. Fix the following data:*

- *a smooth map $f_0 \colon M \to N$,*
- *a closed subset $C \subset M$ such that $f_0 \pitchfork Z$ and $\partial f_0 \pitchfork Z$ on an open neighbourhood U of C,*
- *a closed subset $D \subset M$ and open neighbourhood $V \subset M$ containing $D \backslash U$.*

Then there is an open neighbourhood $U' \subset M$ of $C \cup D$, as well as an $r \geqslant 0$ and a smooth map $F \colon M \times \mathbb{R}^r \to N$ with $F|_{M \times \{0\}} = f_0$ such that

(i) $F|_{M \backslash V \times \{s\}} = f_0|_{M \backslash V}$ for all $s \in \mathbb{R}^r$,

(ii) for each $\epsilon > 0$ there exists an $s \in \mathbb{R}^r$ such that $F|_{U' \times \{s\}}$ and $\partial F|_{U' \times \{s\}}$ are transverse to Z.

As preparation, we construct a smooth function which controls where we manipulate f_0:

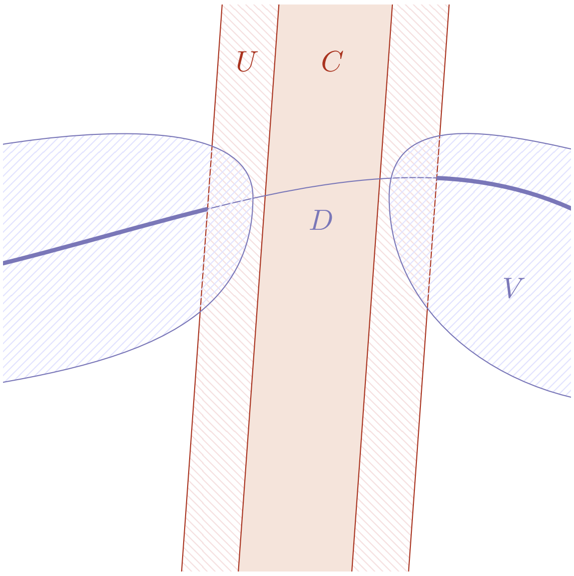


Figure 15.1 The data in Theorem 15.1.1. Eventually U' will be an open neighbourhood of $C \cup D$ inside $U \cup V$.

Lemma 15.1.2. *Suppose we are given closed subsets $C, D \subset M$, an open neighbourhood $U \subset M$ of C and an open neighbourhood $V \subset M$ of $D \backslash U$. Then there exists a smooth function $\gamma\colon M \to [0,1]$ with the following properties:*

- *it has support in V,*
- *is 0 on an open neighbourhood of C, and*
- *is 1 on an open neighbourhood of $D \backslash U$.*

Proof. Take a partition of unity subordinate to $V \backslash C$, U, and $M \backslash (C \cup D)$; we call them $\eta_{V \backslash C}$, η_U and $\eta_{M \backslash (C \cup D)}$. The function $\eta_{V \backslash C}\colon M \to [0,1]$ is the desired γ. By construction, it has support in $V \backslash C \subset V$. Both $\mathrm{supp}(\eta_U)$ and $\mathrm{supp}(\eta_{M \backslash (C \cup D)})$ are closed subsets not containing $D \backslash U$, so the complement of their union contains an open neighbourhood of $D \backslash U$; necessarily $\eta_{V \backslash C} = 1$ there. Similarly, only U contains C so $\eta_U = 1$ on C, and hence $\eta_{V \backslash C} = 0$ on C. □

The proof is now a small variation on the proof that maps transverse to Z are generic, using γ to control the size of deformations.

Proof of Theorem 15.1.1. Embed N into $\mathbb{R}^r$ and take a regular neighbourhood $\pi_N\colon U \to N$. We can find a smooth function $\epsilon\colon N \to (0, \infty)$ such that for each $p' \in N$ and $x \in \mathbb{R}^r$ satisfying $||x|| < \epsilon(p')$ we have $p' + x \in U$. Then we define the smooth map

$$\begin{aligned} F\colon M \times \mathbb{R}^r &\longrightarrow N \\ (p, s) &\longmapsto \pi_N \left(f_0(p) + \gamma(p)\epsilon(f_0(p)) \frac{s}{1 + ||s||^2} \right). \end{aligned}$$

By construction $F|_{M\times\{0\}} = \pi_N \circ f_0 = f_0$, because π_N is the identity on N. Furthermore $f_s = f_0$ on the complement of $V' := \gamma^{-1}((0,1]) \subset V$.

When we fix $p \in V'$ we get a submersion, and the argument of the previous lecture tells us that for a dense set of $s \in \mathbb{R}^r$, we have that f_s and ∂f_s are transverse to Z at $p \in \gamma^{-1}((0,1])$. Furthermore, f_0 and ∂f_0 were already transverse to Z at $p \in U$, and since an open neighbourhood $W \subset U$ of C is contained in $M\backslash V'$, the same is true for f_s and ∂f_s at $p \in W$. We conclude that f_s and ∂f_s are transverse to Z at $p \in V' \cup W$. This is an open neighbourhood $(D\backslash U) \cup C$. Finally, if we take s small enough, the stability of transverse maps will guarantee f_s is transverse to Z an open neighbourhood W' of a closed subset D' of D contained in U and satisfying $C \cup D \subset V' \cup W \cup W'$. □

To apply this result, it is helpful to know that f_0 and ∂f_0 are transverse to Z on an open neighbourhood U of C if and only if they are transverse to Z on C. One direction is obvious, the other holds if Z is closed:

Lemma 15.1.3. *If Z is closed and f_0 and ∂f_0 are transverse to Z on a closed subset $C \subset M$, then there exists an open neighbourhood U of C such that f_0 and ∂f_0 are transverse to Z.*

The idea is essentially the same as the stability of maps transverse to C.

Proof. We prove that such an open neighbourhood exists for each $p \in C$. If $p \notin f_0^{-1}(Z)$ then $M\backslash f_0^{-1}(Z)$ works because Z is closed. If $p \in f_0^{-1}(Z)$, pick a local parametrization $\phi\colon \mathbb{R}^{k'} \supset U' \to V' \subset N$ of Z near $f_0(x)$. If Z is codimension r, $Z \cap V' = \phi(\{0\} \times \mathbb{R}^{k'-r})$. Then f_0 is transverse to $Z \cap V'$ at p' if and only if the derivative at p' of $\pi_r \circ \phi^{-1} \circ f_0$ is surjective. Because surjective linear maps are open, if this is true at p then it must be true for all p' in an open neighbourhood of p. □

Corollary 15.1.4. *Suppose M, N, Z are all without boundary, M compact. If $f_0, f_1\colon M \to N$ are homotopic and both transverse to Z, then there exists a homotopy $H\colon M \times [0,1] \to N$ from f_0 to f_1 which is transverse to Z.*

Proof. Apply Theorem 15.1.1 with f_0 a given homotopy $\tilde{H}\colon M \times [0,1] \to N$, $C = M \times \{0,1\}$ and $D = M \times [0,1]$. The open neighbourhood U is provided by Lemma 15.1.3 and the open neighbourhood V is an open subset of $M \times (0,1)$ containing $M \times [0,1]\backslash U$. Pick an $s \in R^r$ such that $F|_{M\times[0,1]\times\{s\}}$ and $\partial F|_{M\times[0,1]\times\{s\}}$ are transverse to Z. Then $F|_{M\times[0,1]\times\{s\}}$ is the desired homotopy H. □

15.2 Mod 2 intersection theory

Suppose that $Y, Z \subset M$ are compact submanifolds and that $\dim(Y) + \dim(Z) = \dim(M)$. If $Y \pitchfork Z$, then $Y \cap Z$ is a compact 0-dimensional submanifold and hence a finite number of points. If Y is not transverse to Z, we know that we can make it so by a homotopy or even an isotopy. However, the number of points in the intersection make depend on the way we make Y transverse to Z, see Figure 15.2. However, a bit of experimentation suggests that whenever we change the number of intersection points, we either add or remove two points; the number of intersection points mod 2 might be independent of the transverse perturbation! Let us prove this in a bit more generality:

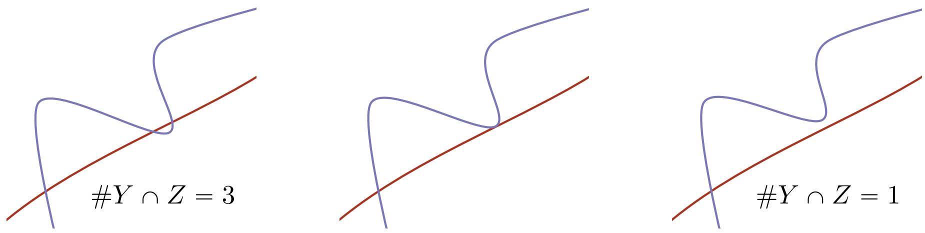


Figure 15.2 Two transverse perturbations with a different number of intersection points.

Definition 15.2.1. Let Y be a compact manifold, M be a manifold, and $Z \subset M$ be a submanifold, all without boundary and satisfying $\dim(Y) + \dim(Z) = \dim(M)$. Let $f_0 \colon Y \to M$ be a smooth map, then the *mod 2 intersection number* $I_2(f_0, Z)$ of f_0 with Z is defined as follows: take f_1 homotopic to f_0 with $f_1 \pitchfork Z$, and set

$$I_2(f_0, Z) := \# f_1^{-1}(Z) \pmod 2.$$

Notation 15.2.2. If f_0 is the inclusion of Y as a submanifold, we shall use the notation $I_2(Y, Z) := I_2(f_0, Z)$.

Lemma 15.2.3. *The number $I_2(f_0, Z) \in \mathbb{Z}/2$ is well-defined.*

Proof. Suppose that f_1 and f_1' are two different smooth maps homotopic to f_0 and transverse to Z. Since homotopy is an equivalence relation, f_1 is homotopic to f_1'. Then Corollary 15.1.4 provides a homotopy $H \colon Y \times [0,1] \to M$ from f_1 to f_1' which is transverse to Z. This means that $H^{-1}(Z)$ is a 1-dimensional submanifold of $Y \times [0,1]$ with boundary

$$\partial H^{-1}(Z) = (\partial H)^{-1}(Z) = (f_1^{-1}(Z) \times \{0\}) \cup ((f_1')^{-1}(Z) \times \{0\}).$$

It is compact because $Y \times [0,1]$ is compact. Since $\#\partial H^{-1}(Z)$ is even by the classification of compact 1-dimensional manifolds, we see that

$$\# f_1^{-1}(Z) + \#(f_1')^{-1}(Z) = \#\partial H^{-1}(Z) \equiv 0 \pmod 2. \qquad \square$$

Example 15.2.4. If $M = \mathbb{R}^n$, then $I_2(f, Z)$ vanishes when $\dim(Y) > 0$. To see this, observe that $M \backslash Z$ is non-empty and open, and hence contains a ball. By composing f with translation and scaling we can homotope f so that its image lies in this little ball and hence disjoint from Z.

Example 15.2.5. Let M be the Moebius strip, $Y = Z$ the central circle. Then $I_2(Y, Z) = 1$ because it is easy to find a small perturbation of Y which makes it intersect Z transversally in a single point.

Here are some basic properties of this invariant of smooth maps $Y \to M$.

Proposition 15.2.6. *The mod 2 intersection number has the following properties:*

(i) If $f, g\colon Y \to M$ are homotopic then $I_2(f, Z) = I_2(g, Z)$.

(ii) If $f\colon Y \to M$ is homotopic to a constant map and $\dim(Y) > 0$, then $I_2(f, Z) = 0$.

(iii) If $Y = \partial W$ for a compact manifold W and $f\colon \partial W \to M$ extends to a smooth map $W \to M$ then $I_2(f, Z) = 0$.

(iv) If we have a pair of smooth maps $f\colon X \to Y$, $g\colon Y \to M$ with X compact, $\dim(X) + \dim(Z) = \dim(M)$, and g transverse to Z, then $I_2(f, g^{-1}(Z)) = I_2(g \circ f, Z)$.

Proof. Part (i) follows from the definitions and the fact that homotopy is an equivalence relation. Part (ii) follows because such an f is homotopic to a map disjoint from Z. Part (iii) follows from the fact that we may assume f transverse to Z and then the extension can be also chosen transverse to Z. In this case $f^{-1}(Z)$ is the boundary of a compact 1-dimensional manifold and must be an even number of points. Part (iv) follows by noting that we may assume that f is transverse to $g^{-1}(Z)$ and then both intersection numbers count the same set. □

15.3 First applications of mod 2 intersection theory

We now give some easy applications of intersection theory, leaving more advanced ones to the next lecture.

15.3.1 Contractible compact manifolds

Let's start with an easy consequence. Recall that a manifold is contractible if its identity is homotopic to a constant map.

Proposition 15.3.1. *The point is the only contractible compact manifold (without boundary).*

Proof. Suppose Y is contractible but not a point. Then Proposition 15.2.6 (ii) applied to $\mathrm{id}\colon Y \to Y$ implies $1 = I_2(\mathrm{id}, \{p\}) = 0$ for any $p \in Y$, an obvious contradiction. □

Remark 15.3.2. As the Whitehead manifold from the additional examples shows, this is false without the compactness assumption.

15.3.2 The mod 2 degree of maps

When $\dim(Y) = \dim(M)$ and M is connected, we can define:

Definition 15.3.3. The *mod 2 degree* $\deg_2(f)$ of a smooth map $f\colon Y \to M$ is given by $I_2(f, \{p\})$ for some $p \in M$.

Lemma 15.3.4. *This is well-defined.*

Proof. We claim that $p \mapsto I_2(f, \{p\})$ is locally constant. Indeed, we may assume that f is transverse to $\{p\}$. Then by the inverse function theorem and the fact that Y is compact, there exists an open neighbourhood U of p such that $f^{-1}(U)$ is a finite disjoint union $\bigsqcup_{i=1}^{k} V_i$ with $p|_{V_i} \colon V_i \to U$ a diffeomorphism. This means that the number of points in the pre-image of f is locally constant, hence so is this number modulo 2. $\square$

Example 15.3.5. The identity map $\mathrm{id}_M \colon M \to M$ has $\deg_2(\mathrm{id}_M) = 1$.

Example 15.3.6. More generally, if $\phi \colon Y \to M$ is a diffeomorphism, then it is transverse to all points in M and the pre-image consists of a single point, so $\deg_2(\phi) = 1$.

Example 15.3.7. If $q \colon E \to B$ is a covering map of degree d with B connected and E compact, then $\deg_2(f) \equiv d \pmod 2$.

We can translate the properties of Proposition 15.2.6 into properties for $\deg_2$:

Proposition 15.3.8. *Suppose Y is compact,* $\dim(Y) = \dim(M)$*, and M is connected, then the mod 2 degree has the following properties:*

(i) If $f, g \colon Y \to M$ are homotopic then $\deg_2(f) = \deg_2(g)$*.*

(ii) If $f \colon Y \to M$ is homotopic to a constant map and $\dim(Y) > 0$ *then* $\deg_2(f) = 0$*.*

(iii) If $Y = \partial W$ for a compact manifold W and $f \colon \partial W \to M$ extends to a smooth map $W \to M$ then $\deg_2(f) = 0$*.*

(iv) If we have a pair of smooth maps $f \colon X \to Y$, $g \colon Y \to M$ with X and Y compact, Y and M connected and $\dim(X) = \dim(Y) = \dim(M)$*, then* $\deg_2(g \circ f) = \deg_2(g) \cdot \deg_2(f)$*.*

Proof. Only (iv) is not obvious. By homotoping g we can make it transverse to $p \in M$, and by homotoping f we can make it transverse to $g^{-1}(p) \subset Y$. Then $g \circ f$ is transverse to p and

$$\begin{aligned}
\deg_2(g \circ f) &= \#(g \circ f)^{-1}(p) \\
&= \#f^{-1}(g^{-1}(p)) \\
&= \sum_{q \in g^{-1}(p)} \#f^{-1}(q) \\
&\equiv \#g^{-1}(p) \cdot \deg_2(f) \\
&= \deg_2(g) \cdot \deg_2(f),
\end{aligned}$$

where we have used that all values $\#f^{-1}(q) \pmod 2$ are equal to $\deg_2(f)$ by the argument used to prove that $\deg_2$ is well-defined (this uses that Y is connected). $\square$

15.3.3 Winding numbers

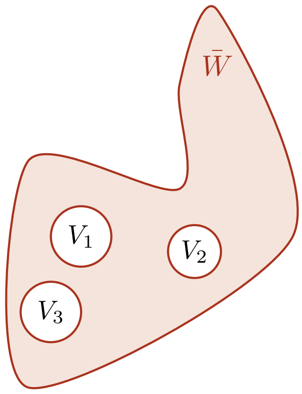
If M is compact manifold of dimension k and $f\colon M \to \mathbb{R}^{k+1}$ is a smooth map, then for $x \notin \mathrm{im}(f)$ we can define a smooth map

$$\begin{aligned} w_{f,z}\colon M &\longrightarrow S^k \\ x &\longmapsto \frac{f(x)-z}{||f(x)-z||} \end{aligned}$$

and then let define the *mod 2 winding number* $W_2(f,z)$ of f around z to be $\deg_2(w_{f,z})$. It only depends on the connected component of $\mathbb{R}^{k+1}\backslash\mathrm{im}(f)$ containing z.

If $M = \partial W$ with W compact and f extends to a smooth map $F\colon W \to \mathbb{R}^{k+1}$, we can often compute $W_2(f,z)$ in terms of F:

Proposition 15.3.9. $W_2(f,z) \equiv I_2(F,z) \pmod 2$.



Proof. It suffices to prove that if z is a regular value of F then $W_2(f,z) = \#F^{-1}(z)$. Because z is a regular value, we can find a small open disk U around z avoiding $f(\partial W)$, such that $f^{-1}(U)$ is a finite disjoint union $\bigsqcup_{i=1}^r V_i$ with $p|_{V_i}\colon V_i \to U$ a diffeomorphism, with $r = \#F^{-1}(z)$. Then $\overline{W} := W\backslash\bigsqcup_{i=1}^r V_i$ is another compact manifold with boundary and F restricts to a smooth map $\overline{F} := F|_{\overline{W}}\colon \overline{W} \to \mathbb{R}^{k+1}$.

Since this avoids z, there is a smooth map

$$\begin{aligned} \overline{F}\colon \overline{W} &\longrightarrow S^k \\ x &\longmapsto \frac{F(x)-z}{||F(x)-z||} \end{aligned}$$

and by Sard's theorem we can find a $p \in S^k$ such that $\overline{F}$ and $\partial\overline{F}$ are transverse to p. Hence $\bar{F}^{-1}(p)$ is one-dimensional compact submanifold on $\bar{W}$, and its boundary is an even number of points. This implies that

$$\begin{aligned} 0 &\equiv \#\partial\overline{F}^{-1}(p) \pmod 2 \\ &= \#w_{f,z}^{-1}(p) + \sum_{i=1}^{r} \#w_{F|_{\partial V_i},z}^{-1}(p) \\ &= W_2(f,z) + \sum_{i=1}^{r} W_2(F|_{\partial V_i}, z). \end{aligned}$$

Thus we may as well compute each $W_2(F|_{\partial V_i}, z)$.

Since $F|_{\partial V_i}$ is a diffeomorphism, given by the composition of the inclusion $i\colon \partial U \hookrightarrow \mathbb{R}^{k+1}$ with a diffeomorphism, each of these is equal to $W_2(i,z)$. Since $w_{i,z}\colon \partial U \to S^k$ is given by a composition of translation and scaling, it is a diffeomorphism; by Example 15.3.6 $W_2(i,z) = 1$. We conclude that $W_2(f,z) \equiv r \pmod 2$, as desired. □

15.4 Problems

Problem 39 (Spheres are not products)**.** Let M and N be compact connected smooth manifolds of dimension k and $n-k$ respectively, and suppose that $k > 0$ and $n - k > 0$. Fixing $q_0 \in N$ there is an inclusion $i_{q_0}\colon M \to M \times N$ given by $p \mapsto (p, q_0)$.

(a) Prove that if S^n is diffeomorphic to $M \times N$ then i_{q_0} is homotopic to a constant map.

(b) Prove that S^n is not diffeomorphic to $M \times N$ using intersection theory.

Problem 40 (Bordism-invariance of intersection numbers)**.** A *bordism* is a compact manifold W with boundary ∂W divided into two submanifolds $\partial_0 W$ and $\partial_1 W$. Suppose that we have a smooth map

$$F\colon W \longrightarrow M$$

and a smooth submanifold $Z \subset M$ so that $\dim(Z) + \dim(W) = \dim(M) + 1$. Prove that $I_2(F|_{\partial_0 W}, Z) = I_2(F|_{\partial_1 W}, Z)$.

Chapter 16

Two applications of mod 2 intersection theory

We continue with our discussion of mod 2 intersection theory and its applications. This includes some applications from [Mat03] and Section 2.§5 of [GP10].

16.1 The Borsuk–Ulam theorem

Recall that if M is compact smooth manifold of dimension k and $f\colon M \to \mathbb{R}^{k+1}$ is a smooth map, then for $x \notin \operatorname{im}(f)$ we can define a smooth map

$$\begin{aligned} w_{f,z}\colon M &\longrightarrow S^k \\ x &\longmapsto \frac{f(x)-z}{||f(x)-z||}. \end{aligned}$$

The *mod 2 winding number* $W_2(f,z)$ of f around z is then $\deg_2(w_{f,z})$. As an application of mod 2 winding numbers we will prove the *Borsuk–Ulam theorem*. Before doing so, let us start with an easier example of how conditions on a smooth map constrain its winding number:

Proposition 16.1.1. *If a smooth map $f\colon S^k \to \mathbb{R}^{k+1}\backslash\{0\}$ satisfies $f(-x) = f(x)$, then $W_2(f,0) = 0$.*

Proof. We start with the observation that f is homotopic as a smooth map $S^k \to \mathbb{R}^{k+1}\backslash\{0\}$ to $f/||f||$ by $(p,t) \mapsto f/(1-t+t||f||)$, and that this satisfies the same symmetry condition. Hence, without loss of generality we are dealing with a smooth map $f\colon S^k \to S^k$. Then $w_{f,0} = f$, and we are equivalently proving a result about the degree of f. The symmetry condition implies that f factors as

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "q"'] & S^n \\ \mathbb{R}P^n \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Since q is a double cover, $\deg_2(q) \equiv 0 \pmod 2$, and we get $\deg_2(f) = \deg_2(q)\deg_2(\bar{f}) = 0$ as well. $\square$

Theorem 16.1.2 (Borsuk–Ulam)**.** *If a smooth map $f\colon S^k \to \mathbb{R}^{k+1}\backslash\{0\}$ satisfies $f(-x) = -f(x)$, then $W_2(f,0) = 1$.*

Proof. As above, without loss of generality we may assume we are dealing with a smooth map $f\colon S^k \to S^k$, and we may use $W_2(f,0)$ and $\deg_2(f)$ interchangeably. The proof is by induction over k, with the formal properties of the winding number playing a major role in the induction step.

We start with the initial case $k = 0$. Then $f\colon S^0 \to S^0$ is either the identity $\mathrm{id}\colon S^0 \to S^0$ or $-\mathrm{id}$. Both id and $-\mathrm{id}$ are diffeomorphisms and hence have degree 1.

For the induction step, we assume the result is true for $k-1$ and prove it for k. We are given a smooth map $f\colon S^k \to S^k$ satisfying $f(-x) = -f(x)$, and define $g := f|_{S^{k-1}}$ which also satisfies $g(-x) = -g(x)$. By Sard's theorem there exists an $a \in \mathrm{int}(S^k_+)$, with $S^k_+ = S^k \cap [0,\infty) \times \mathbb{R}^k$ the upper hemisphere, which is a regular value of both f and g. By symmetry $-a$ is also a regular value of both f and g. We can use this to rewrite $\deg_2(f)$:

$$\deg_2(f) \equiv \#f^{-1}(a) = \frac{1}{2}(\#f^{-1}(a) + \#f^{-1}(-a)).$$

To apply the induction hypothesis we want to go from S^k to something diffeomorphic to $\mathbb{R}^k$. Let $\pi\colon \mathbb{R}^{k+1} \to a^\perp \cong \mathbb{R}^k$ by the orthogonal projection. That $g \pitchfork \{a,-a\}$ means that the image of g is disjoint from a and $-a$ and hence $\pi \circ g$ avoids 0. Since furthermore $f \pitchfork \{a,-a\}$, $\pi \circ f|_{S^k_+}$ is transverse to 0 and we have

$$\#(\pi \circ f|_{S^k_+})^{-1}(0) = \#(f|_{S^k_+})^{-1}(a) + \#(f|_{S^k_+})^{-1}(-a) = \frac{1}{2}(\#f^{-1}(a) + \#f^{-1}(-a)).$$

This means that $\deg_2(f) \equiv \#(\pi \circ f|_{S^k_+})^{-1}(0)$.

Now recall that by a previous proposition about computing winding numbers using extension, with $W = S^k_+$, $F = f|_{S^k_+}$ and $z = 0$, we have that

$$\#(\pi \circ f|_{S^k_+})^{-1}(0) \equiv W_2(\pi \circ g, 0) \pmod 2.$$

As $W_2(\pi \circ g, 0) = \deg_2(\pi \circ g)$ and since π is linear, $\pi \circ g(-x) = -\pi \circ g(x)$ so that the inductive hypothesis applies and thus $\deg_2(\pi \circ g) = 1$. □

16.1.1 Applications of the Borsuk–Ulam theorem

In this section deduces several famous consequences of Theorem 16.1.2.

Corollary 16.1.3. *If a smooth map $f\colon S^k \to \mathbb{R}^{k+1}\backslash\{0\}$ satisfies $f(-x) = -f(x)$, then f intersects every line through the origin at least once.*

Proof. If the image of f does not intersect ℓ, we compute that $W_2(f,0) = 0$ using an element $p \in S^k \cap \ell$, contradicting Theorem 16.1.2. □

This corollary can be restated in a number of equivalent forms. We purposefully are a bit whether the maps are smooth or not; by an application of the Stone–Weierstrass approximation theorem the results for smooth maps imply those for continuous maps.

Theorem 16.1.4. *The following are equivalent:*

(i) If $f\colon S^k \to \mathbb{R}^{k+1}\backslash\{0\}$ satisfies $f(-x) = -f(x)$, then f intersects every line through the origin at least once.

(ii) If $g\colon S^k \to \mathbb{R}^k$ satisfies $g(-x) = -g(x)$, then g has a zero.

(iii) Every $h\colon S^k \to \mathbb{R}^k$ has an x such that $h(x) = h(-x)$.

(iv) There is no $F\colon S^k \to S^{k-1}$ satisfying $F(-x) = -F(x)$.

(v) There is no $G\colon D^k \to S^{k-1}$ satisfying $G(-x) = -G(x)$ for $x \in \partial D^k$.

Proof.

- We start with (i) $\Rightarrow$ (ii). If g has no zero then

$$\begin{aligned} f\colon S^k &\longrightarrow \mathbb{R}^{k+1}\backslash\{0\} \\ x &\longmapsto (g(x), 0) \end{aligned}$$

  avoids the x_{k+1}-axis, contradicting (i).
- For (ii) $\Rightarrow$ (i), if f avoids ℓ and $\pi\colon \mathbb{R}^{k+1} \to \mathbb{R}^k$, then taking $g(x) = \pi \circ f(x)$ would contradict (ii).
- For (ii) $\Rightarrow$ (iii), take $g(x) = h(x) - h(-x)$.
- For (iii) $\Rightarrow$ (ii), there is an x such that $-g(x) = g(-x) = g(x)$ so $g(x) = 0$.
- For (ii) $\Leftrightarrow$ (iv), we just normalize.
- For (iv) $\Rightarrow$ (v), use that from such an G we could produce an F by picking a diffeomorphism $\phi\colon S^k_+ \to D^k$ that is the identity on the boundary and setting $F(x) = G(\phi(x))$ for $x \in S^k_+$ and $F(x) = -G(\phi(-x))$ for $x \in \mathrm{int}(S^k_-)$.
- For (v) $\Rightarrow$ (iv) use that from such an F we could produce a G by taking $F|_{S^k_+} \circ \phi^{-1}\colon D^k \to S^{k-1}$. □

Example 16.1.5. Theorem 16.1.4 (v) gives another proof that there is no continuous map $D^k \to \partial D^k$ which is the identity on ∂D^k, a special case of Hirsch's generalisation of the Brouwer fixed point theorem.

Part (iii) of Theorem 16.1.4 has several famous geometric applications; see [Mat03] for even more:

Corollary 16.1.6 (Lusternik–Schnirelmann)**.** *If $U_0, \ldots, U_k$ is an open cover of S^k then there is an $i \in \{0, \ldots, k\}$ such that $U_i \cap (-U_i) \neq \varnothing$.*

Here $(-U_i)$ is of course the set $\{z \in S^k \mid -z \in U_i\}$.

Proof. We first prove that if $C_0, \ldots, C_k$ is a cover of S^k by closed sets then there is an i such that $C_i \cap (-C_i) \neq \varnothing$. Consider the continuous function

$$\begin{aligned} g\colon S^k &\longrightarrow \mathbb{R}^k \\ x &\longmapsto \big((d(x, C_1), \ldots, d(x, C_n)\big) \end{aligned}$$

with $d(-,-)$ the ordinary Euclidean metric on $\mathbb{R}^{k+1}$. By Theorem 16.1.4 (iii) there must be an x such that $g(x) = g(-x)$. If the ith entry of $g(x)$ is 0, then

$x, -x \in C_i$. If none of the entries of $g(x)$ are 0, then $x, -x \notin \bigcup_{i=1}^n C_i$ and hence $x, -x \in C_{n+1}$.

The version for open covers follows using the fact that a partition of unity subordinate to an open cover $U_0, \dots, U_k$ of S^k such that all $U_i \cap (-U_i) = \varnothing$ for all i, provides a closed cover $C_i := \operatorname{supp}(\eta_i)$ of S^k by closed subsets with the same property. $\square$

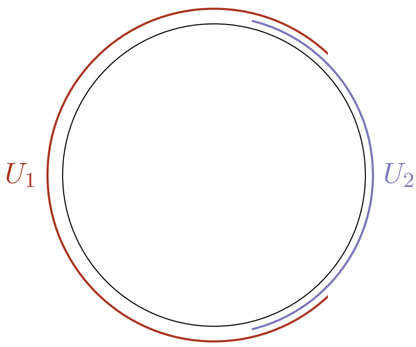


Figure 16.1 A cover of S^1 by two open subsets. The open subset U_1 contains two antipodal points.

Corollary 16.1.7 (Ham–Sandwich)**.** *Let $M_1, \dots, M_n$ be bounded measurable subsets of $\mathbb{R}^n$ of positive measure. Then there exists an affine hyperplane $h \subset \mathbb{R}^n$ such that each of both of the half-spaces $h^{\pm}$ bounded by h we have $\mu(M_i \cap h^+) = \mu(M_i \cap h^-)$ for all $1 \leqslant i \leqslant n$.*

Proof. Without loss of generality $M_1, \dots, M_n \subset B_1(0)$. For each $x \in S^k$ we can define a subspace h_x^+ when $x_{k+1} \neq \pm 1$, $h_x^+ := \{(v_1, \dots, v_k) \in \mathbb{R}^k \mid \sum_{i=1}^k x_i v_i \geqslant x_{k+1}\}$. Note that if $x = e_{k+1}$ we have $h_x^+ = \varnothing$ and that if $x = -e_{k+1}$ we have $h_x^+ = \mathbb{R}^k$.

We define a function

$$\begin{aligned} g\colon S^k &\longrightarrow \mathbb{R}^k \\ x &\longmapsto (\mu(M_1 \cap h_x^+), \dots, \mu(M_k \cap h_x^+)). \end{aligned}$$

We will leave it to Theorem 3.1.1 in [Mat03] the proof that this is continuous. By Theorem 16.1.4 (iii) there must be an x such that $g(x) = g(-x)$. Since h_{-x}^+ is h_x^-, this means that $(\mu(M_1 \cap h_x^+), \dots, \mu(M_k \cap h_x^+)) = (\mu(M_1 \cap h_x^-), \dots, \mu(M_k \cap h_x^-))$. $\square$

In other words, you can slice even an irregular sandwich with a slice of ham and a slice of cheese, such that the bread, ham and cheese are all divided in half.

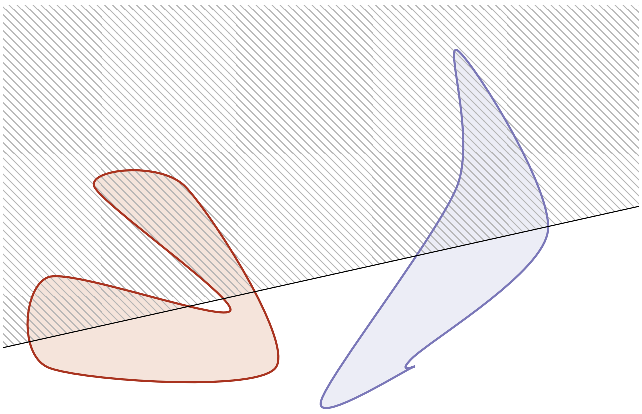


Figure 16.2 There exists a half-plane which contains half of both the red and the blue figure (this is probably not it).

16.2 The Jordan–Brouwer separation theorem

16.2.1 Its proof

One can also use the ideas behind mod 2 intersection theory to deduce the famous Jordan–Brouwer separation theorem. Section 2.§5 of [GP10] deduces it from winding numbers, but I think this direct proof is clearer.

Theorem 16.2.1. *If $Z \subset S^n$ is a compact connected non-empty submanifold of dimension $n-1$, then $S^n \backslash Z$ is a disjoint union of two connected open subsets, each of which has closure a compact submanifold with boundary Z.*

By removing a point from $S^n \backslash Z$, we reduce to the case $\mathbb{R}^n \backslash Z$; in this case we only get the second claim for one of the both components but since we could have removed any point the same is true for the other component.

Example 16.2.2. In dimension 2 we are saying that a curve in the plane divides it into two pieces. See `https://www.maths.ed.ac.uk/~v1ranick/papers/jordan-revised` for some examples of complicated curves if you think this is obviously true.

Proof of Theorem 16.2.1. Pick an $x_0 \in \mathbb{R}^n \backslash Z$. To simplify very end of the proof, we will assume that x_0 lies outside some closed disk around the origin containing the compact subset Z.

We claim that there is locally constant assignment $d\colon \mathbb{R}^n \backslash Z \to \mathbb{Z}/2$, given at $x \in \mathbb{R}^n \backslash Z$ by picking a smooth path γ from x to x_0 which is transverse to Z and taking $d(x)$ to be $\#\gamma^{-1}(Z) \pmod 2$. Let us prove that this is well-defined.

To show that such a γ exists, observe that for each $x \in \mathbb{R}^n \backslash Z$ there is an open ball $B_\epsilon(x) \subset \mathbb{R}^n \backslash Z$ around x. We define a smooth map

$$\begin{aligned} F\colon [0,1] \times B_\epsilon(x) &\longrightarrow \mathbb{R}^n \\ (t,y) &\longmapsto ty + (1-t)x_0, \end{aligned}$$

which is visibly a submersion when restricted to fixed $t \in [0,1]$, so $F \pitchfork Z$, $\partial F \pitchfork Z$ (in fact, ∂F avoids Z all-together). By Theorem 14.1.3 there exists a dense set of $y \in B_\epsilon(x)$ such that $F|_{[0,1]\times\{y\}} \pitchfork Z$. Now we let γ be the concatenation of the linear path from x to y and $F|_{[0,1]\times\{y\}}$.

We claim that $\#\gamma^{-1}(Z)$ is independent of the choice of γ. Given two choices γ, γ', consider the map

$$\begin{aligned} G\colon (0,1)\times[0,1] &\longrightarrow \mathbb{R}^n \\ (t,s) &\longmapsto s\gamma(t) + (1-s)\gamma'(t), \end{aligned}$$

This is transverse to Z on an open neighbourhood of the closed subset

$$C = (0,\epsilon] \times [0,1] \cup [1-\epsilon,1) \times [0,1] \cup (0,1) \times \{0,1$$

so by the strongly relative transversality theorem, Theorem 15.1.1, there is a homotopic map which coincides with G near C and is tranverse to Z. Then $G^{-1}(Z)$ is a 1-dimensional submanifold, which is contained in some compact subset of $(0,1)\times[0,1]$, since G avoids Z on $(0,\epsilon')\times[0,1] \cup (1-\epsilon',1)\times[0,1]$ for some $\epsilon' > 0$. Hence it is a compact 1-dimensional submanifold, and hence its boundary contains an even number of points by Theorem 13.3.1. This implies that the difference between $\#\gamma^{-1}(Z)$ and $\#(\gamma')^{-1}(Z)$ is even.

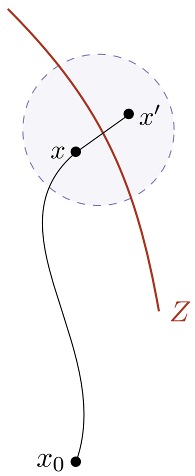


Figure 16.3 Proving that $d\colon \mathbb{R}^n \backslash Z \to \mathbb{Z}/2$ takes both values.

By construction, this function d is constant on connected components. To see it takes both values, look at a chart exhibiting Z as a submanifold, i.e. a diffeomorphism $\phi\colon \mathbb{R}^n \supset U \to V \subset \mathbb{R}^n$ such that $\phi^{-1}(Z \cap V) = (\{0\}\times\mathbb{R}^{n-1}) \cap U$. Suppose that d takes value 0 on say $\phi(((-\infty,0)\times\mathbb{R}^{n-1}) \cap U)$. Then by concatenating γ with the image under ϕ of a straight line segment connecting a point x in $(-\infty,0)\times\mathbb{R}^{n-1}$ with a point x' in $(0,\infty)\times\mathbb{R}^{n-1}$ we see that d takes value 1 on $\phi(((0,\infty)\times\mathbb{R}^{n-1}) \cap U)$. That is, crossing Z changes d by 1. We conclude that $\mathbb{R}^n\backslash Z$ has at least 2 connected components.

To show it has exactly two connected components we need to use that Z is connected. For any fixed $x \in \mathbb{R}^n \backslash Z$, let $V \subset Z$ be the subset of points $z \in Z$ such that any open neighbourhood U of z in $\mathbb{R}^n$ contains a point which has a path to x avoiding Z. This is closed and open by looking at charts exhibiting Z as a submanifold, and is non-empty by looking at a point in Z closest to x. Thus, V is union of connected components of Z and hence all of Z.

Now let us look at opposite sides of Z in a fixed chart; by the above argument, each $x \in \mathbb{R}^n \backslash Z$ can be connected to a point within this chart by a path avoiding Z. This includes x_0 and so can be used to divide the points of $\mathbb{R}^n \backslash Z$ into two path-components (possibly empty); those that connect to x_0 and those that do not. Hence $\mathbb{R}^n \backslash Z$ has at most two connected components and hence exactly two, given by $d^{-1}(0)$ and $d^{-1}(1)$ respectively.

To see that the closure of $d^{-1}(0)$ is a manifold with boundary we need to find charts near boundary points. Note that for each local trivialization of Z, exactly one of $\phi(((-\infty, 0) \times \mathbb{R}^{n-1}) \cap U)$ and $\phi(((0, \infty) \times \mathbb{R}^{n-1}) \cap U)$ lies in $d^{-1}(0)$, say the latter, and then $\phi|_{([0,\infty) \times \mathbb{R}^{n-1}) \cap U)}$ is the desired chart near the boundary. The same argument applies to $d^{-1}(1)$.

Finally, any points x with $||x|| \geqslant ||x_0||$ can be connected to x_0 by a path avoiding Z, so the closure $d^{-1}(1)$ is bounded and hence compact. □

Let us reflect on the proof. What did we really use about $\mathbb{R}^n$? Only that it is connected and simply-connected. That is, for the definition of d we only need to be able connect x to x_0 by some path γ. To show it is well-defined, we need that any two choices γ and γ' are homotopic relative to their endpoints. Thus, the same proof gives the following generalization of the Jordan-Brouwer separation theorem:

Theorem 16.2.3. *Suppose M is a simply-connected connected compact manifold of dimension n and $Z \subset M$ is a compact connected non-empty submanifold of dimension $n-1$, then $M \backslash Z$ is a disjoint union of two connected open subsets, each of which has closure a compact submanifold with boundary Z.*

16.2.2 The Schoenflies theorem

In particular, if $i\colon S^{k-1} \hookrightarrow S^k$ is a smooth embedding then $i(S^{k-1})$ divides S^k into two connected components, and the closure of each of these is a compact submanifold with boundary. What are these manifolds with boundary? Of course, taking i to be the standard inclusion we get two disks D^k. Can other manifolds appear? The answer is "no" in low dimensions:

Theorem 16.2.4 (Schoenflies, Alexander)**.** *If $k \leqslant 3$, for each embedding $i\colon S^{k-1} \hookrightarrow S^k$ the closures of both components of $S^k \backslash S^{k-1}$ are diffeomorphic to D^k.*

You can find a proof for $k = 3$ in [Hat07, Theorem 3.3], which you should be able to adapt to $k = 2$ without much difficulty.

However, in high dimensions there *can* be. One of the successes of differential topology is the determination of dimensions in which this can happen in terms of other well-studied objects in algebraic topology (the groups of exotic spheres). In particular, in dimension $\leqslant 140$ we have [BHHM17]:

Theorem 16.2.5. *If $5 \leqslant k \leqslant 140$, for each embedding $i\colon S^{k-1} \hookrightarrow S^k$ the closures of both components of $S^k \backslash S^{k-1}$ are diffeomorphic to D^k if and only if $k = 5, 6, 12, 56, 61$.*

There is one dimension remaining for $k \leqslant 140$: $k = 4$. One of the big remaining open questions of manifold theory asks about this case:

Conjecture 16.2.6 (Smooth Schoenflies conjecture in dimension 4)**.** *Given an embedding $i\colon S^3 \hookrightarrow S^4$, the closures of both components of $S^4 \backslash S^3$ are diffeomorphic to D^4.*

16.2.3 Codimension one knots

Just we called (isotopy classes of) embeddings of S^1 in S^3 are knots, we refer to (isotopy classes of) embeddings $S^{k-r} \hookrightarrow S^k$ as *codimension r knots.* The most interesting case is, unsurprisingly, codimension 2. What about codimension 1?

If for each embedding $i\colon S^{k-1} \hookrightarrow S^k$ the closure of one of the components of $S^k \backslash S^{k-1}$ are diffeomorphic to D^k, there exists only one embedding $S^{k-1} \to S^k$ up to isotopy:

Theorem 16.2.7. *If an embedding $i\colon S^{k-1} \hookrightarrow S^k$ has the property that the closure of one of the components of $S^k \backslash S^{k-1}$ is diffeomorphic to D^k, then i is isotopic to the standard inclusion $S^{k-1} \to S^k$.*

It will follow from:

Proposition 16.2.8. *Every embedding $S^{k-1} \hookrightarrow \mathbb{R}^k$ which extends to an embedding $D^k \hookrightarrow \mathbb{R}^k$ is isotopic to either the standard inclusion i, or i composed with a reflection.*

Proof. We prove that every embedding $D^k \hookrightarrow \mathbb{R}^k$ is isotopic to one given by applying invertible linear map $A \in \mathrm{GL}_k(\mathbb{R})$ to D^k. The result follows from the observation that the two different connected components of $\mathrm{GL}_k(\mathbb{R})$ contain the identity and a reflection respectively.

We claim that embeddings $D^k \hookrightarrow \mathbb{R}^k$ up to isotopy are in bijection with injective immersions $\mathbb{R}^k \hookrightarrow \mathbb{R}^k$ up to homotopy through injective immersions. This bijection is given in one direction by the composing with the embedding $i\colon D^k \hookrightarrow \mathbb{R}^k$, and in the other by composing with the injective immersion $h\colon \mathbb{R}^k \hookrightarrow D^k$ given by $z \mapsto \frac{z}{1+||z||^2}$. It is easy to see that $h \circ i$ is isotopic to id_{D^k}, and $i \circ h$ admits an homotopy through injective immersions to $\mathrm{id}_{\mathbb{R}^k}$.

Now apply Lemma 16.2.9, which classifies injective immersions $\mathbb{R}^k \hookrightarrow \mathbb{R}^k$ up to homotopy through injective immersions. ☐

Lemma 16.2.9. *Every injective immersion $f\colon \mathbb{R}^k \hookrightarrow \mathbb{R}^k$ is homotopic through injective immersions to an invertible linear transformation.*

Proof sketch. Identify $[0,1]$ with $[1,\infty]$ and take

$$\begin{aligned} H\colon \mathbb{R}^n \times [1,\infty] &\longrightarrow \mathbb{R}^n \\ (x,t) &\longmapsto \begin{cases} \frac{1}{t}\cdot h(tx) & \text{if } t<\infty, \\ D_0 h(x) & \text{if } t=\infty. \end{cases} \end{aligned}$$

To see that this is smooth at $t=\infty$ apply Taylor's theorem. □

We can now complete the argument:

Proof of Theorem 16.2.7. We may assume $e_{k+1} \in S^k \backslash S^{k-1}$ is not in the image of the extension, and removing this point, we may as well work in $\mathbb{R}^k$. The result follows by observing that the embeddings $S^{k-1} \hookrightarrow S^k$ given by i and i composed with a reflection are isotopic, as the action of $\mathrm{GL}_k(\mathbb{R})$ on S^n extends to an action of $\mathrm{GL}_{k+1}(\mathbb{R})$ and there is an element of $\mathrm{GL}_k(\mathbb{R})$ with determinant $+1$ which acts on i by reflection. □

Thus Theorem 16.2.5 tells us the following about the existence of codimension one knots.

Corollary 16.2.10. *If* $4 \neq k \leqslant 140$ *and* $k = 0,1,2,3,5,6,12,56,61$*, then every embedding* $S^{k-1} \hookrightarrow S^k$ *is isotopic to the standard inclusion.*

16.3 Problems

Problem 41. Use the Jordan–Brouwer separation theorem to prove that if $M \subset \mathbb{R}^k$ is a compact codimension 1 submanifold, then its normal bundle NM is trivial.

Problem 42. Adapt the proof of Lemma 16.2.9 to prove that every diffeomorphism of $\mathbb{R}^k$ is isotopic to an invertible linear transformation.

Chapter 17

Orientations and integral intersection theory

The next part of these lecture will be devoted defining de Rham cohomology, developing computational tools for it, and drawing interesting topological conclusions from it. A prerequisite for some of this material will be the notion of an orientation. We define this today, and give a taste of Chapter 3 of [GP10], which we will not cover in detail in the course.

Convention 17.0.1. All vector spaces are finite-dimensional and over $\mathbb{R}$ unless mentioned otherwise.

17.1 What is an orientation on a manifold?

We start with an intuitive description of orientations, before giving rigorous definitions:

> an orientation of a manifold is "a smooth family of orientations of each of the tangent spaces T_pM."

An *orientation* on a vector space such as T_pM specifies for each of its ordered bases whether it is "positively oriented" or "negatively oriented," with the following requirement: if one ordered basis can be obtained from another by applying an invertible matrix A to each of its vectors, then they are similarly oriented if and only if $\det(A) > 0$. Since $\mathrm{GL}_n(\mathbb{R})$ has two path components, this is equivalent to saying homotopic bases are similarly oriented and reflecting a single basis vector changes the orientation of the basis.

That an orientation depends smoothly on $p \in M$ means that if you move a positively oriented basis around M, it stays positive (and of course the same is true for negatively oriented bases).

Example 17.1.1. For the circle S^1, an orientation is a choice of "positive direction" along the circle. There are two such choices: counterclockwise and clockwise.

Example 17.1.2. The real projective plane $\mathbb{R}P^2$ admits no orientation. Suppose it did, then starting with a basis e_1, e_2 at some point, say positively oriented, we can move it around $\mathbb{R}P^2$ and return to $e_1, -e_2$. This must simultaneously be positively oriented (since moving a basis around shouldn't change how it's oriented) and negatively oriented (since it is obtained from a positively oriented by reflecting a basis vector). This gives a contradiction.

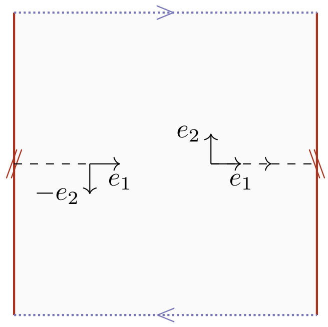


Figure 17.1 Moving a basis around $\mathbb{R}P^2$ can return it with opposite orientation.

You can find more examples in the following table:

orientable	not orientable
spheres S^n	real projective spaces $\mathbb{R}P^{2n}$ $(n \geqslant 1)$
surfaces of genus $g \geqslant 1$	Klein bottle
Lie groups	
Lens spaces	
Poincaré homology sphere	
Complex projective spaces	
Quaternionic projective spaces	
$K3$ surface	
Whitehead manifold	

Example 17.1.3. An LCD display is made from a *nematic crystal*, consisting of long thin filaments. These prefer to be aligned the same way, so locally such a crystal has a *order parameter* given by a direction in $\mathbb{R}^3$. This is an element of $\mathbb{R}P^2$, a non-orientable manifold. [1]

17.2 A recollection of multilinear algebra

Linear algebra concerns not only the study of vector spaces and linear maps between them, but also of multilinear maps with various properties. This is closely related to the study of tensor products and variations thereof.

17.2.1 Tensor products

Definition 17.2.1. A *bilinear map* is a function $b\colon V \times V' \to W$ which is linear in each variable.

Definition 17.2.2. The *tensor product* $V \otimes V'$ is the quotient of the free $\mathbb{R}$-vector space on the set $V \times V'$, whose basis elements we shall denote (v, v'), by the

[1] See http://www.lassp.cornell.edu/sethna/pubPDF/OrderParameters.pdf.

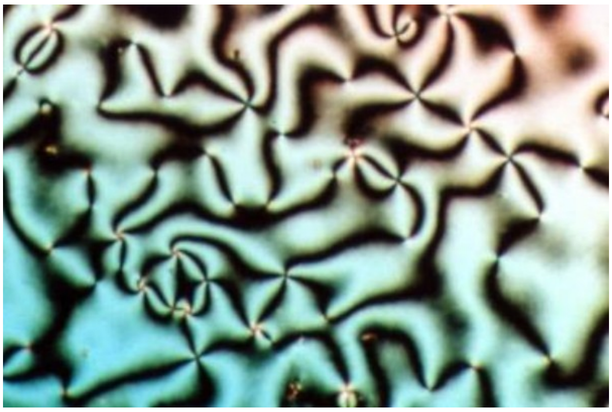


Figure 17.2 An nematic crystal (from https://en.wikipedia.org/wiki/Liquid_crystal).

subspace spanned by the elements

$$((v_1 + v_2), v') - (v_1, v') - (v_2, v'),$$

$$(v, (v_1' + v_2')) - (v, v_1') - (v, v_2'),$$

$$(av, w) - a(v, w),$$

$$(v, aw) - a(v, w).$$

We will denote the equivalence class of (v, w) by $v \otimes w$.

Example 17.2.3. The tensor product $\mathbb{R}^k \otimes \mathbb{R}^l$ has a basis given by $e_i \otimes e_j'$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$.

The relations are designed to make

$$\begin{aligned} b_0 \colon V \times V' &\longrightarrow V \otimes V' \\ (v, v') &\longmapsto v \otimes v' \end{aligned}$$

bilinear. It is in fact the initial bilinear map:

Lemma 17.2.4. *For every bilinear map $b \colon V \times V' \to W$ there is a unique linear map $\beta \colon V \otimes V' \to W$ such that $b = \beta \circ b_0$.*

Proof. There is a unique linear map $\mathbb{R}[V \times V'] \to W$ given by $(v, v') \mapsto b(v, v')$. Since b is bilinear this factors over $V \otimes V'$, determining a linear map $\beta \colon V \otimes V' \to W$ satisfying $b(v, v') = \beta(v \otimes v') = \beta(b_0(v, v'))$. Since $V \otimes V'$ is generated by the elements $b_0(v, v')$, this determines β uniquely. $\square$

Remark 17.2.5. This universal property satisfied by the tensor product determines it uniquely up to linear isomorphism.

There is a similar correspondence of multilinear maps $V_1 \times \cdots \times V_k \to W$ with linear map $V_1 \otimes \cdots \otimes V_k \to W$.

Example 17.2.6. The universal property tells us what tensor product of a single or no vector spaces is. A multilinear map $V \to W$ is just a linear map, so a tensor product of a single vector space V is just V again.

The empty product of sets is a point, because such a product receives a unique map from every other set. A multilinear map from an empty product is hence a map from a point to V, with no condition imposed, so just an element of V. This is the same as a linear map $\mathbb{R} \to V$. Hence an empty tensor product is $\mathbb{R}$ itself.

17.2.2 Alternating multilinear maps

When all vector spaces V_i in the domain of a multilinear map are the same V, we can require additional symmetry properties. Of specific interest to us are the alternating multilinear maps, though the story for symmetric multilinear maps is similar:

Definition 17.2.7. An *alternating multilinear map* is a multilinear map $w\colon V^k \to W$ which satisfies $w(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (-1)^{\epsilon(\sigma)} w(v_1, \ldots, v_k)$ for all $v_1, \ldots, v_k \in V$ and permutations σ of $\{1, \ldots, k\}$. Here $\epsilon(\sigma) \in \mathbb{Z}/2$ is the sign of the permutation.

Example 17.2.8. The sign of a permutation is uniquely determined by demanding it is a homomorphism and it sends a transposition to the unique non-identity element of $\mathbb{Z}/2$.

There is also an initial alternating multilinear map.

Definition 17.2.9. The *kth exterior power* $\Lambda^k V$ is the quotient of $V^{\otimes k}$ by the subspace spanned by the elements

$$v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} - (-1)^{\epsilon(\sigma)} v_1 \otimes \ldots \otimes v_k \text{ with } \sigma \in \Sigma_k.$$

We will denote the image of $v_1 \otimes \cdots \otimes v_k$ by $v_1 \wedge \cdots \wedge v_k$.

Example 17.2.10. $\Lambda^2\mathbb{R}^n$ has a basis $e_i \wedge e_j$ for $1 \leqslant i < j \leqslant n$. It is a well-known mistake to think that every element of an exterior product is of the form $v_1 \wedge v_2$. This is not the case, e.g. $e_1 \wedge e_2 + e_3 \wedge e_4$ can't be written this way.

Example 17.2.11. $\Lambda^0 V$ is the quotient of $(V)^{\otimes 0} = \mathbb{R}$ by the trivial subspace, so is equal to $\mathbb{R}$.

The subspace in Definition 18.2.9 is designed to make

$$\begin{aligned} w_0\colon V^k &\longrightarrow \Lambda^k V \\ (v_1, \ldots, v_k) &\longmapsto v_1 \wedge \cdots \wedge v_k \end{aligned}$$

alternating multilinear. This satisfies:

Lemma 17.2.12. *For every alternating multilinear map $w\colon V^k \to W$ there is a unique linear map $\omega\colon \Lambda^k V \to W$ such that $w = \omega \circ w_0$.*

Remark 17.2.13. This universal property tells us that the map $V^{\otimes k} \to \Lambda^k V$ corresponding to a natural assignment of an alternating multilinear map $w(b) \colon V^k \to W$ to each multilinear map $b \colon V^k \to W$. This is given by anti-symmetrizing:

$$w(b)(v_1, \ldots, v_k) = \frac{1}{n!} \sum_{\sigma \in \Sigma_k} b(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

The construction of $\Lambda^k V$ is natural in V: whenever we have a linear map $A \colon V \to V'$, there is an alternating multilinear map

$$\begin{aligned} V^{\times k} &\longrightarrow \Lambda^k(V') \\ (v_1, \ldots, v_k) &\longmapsto A(v_1) \wedge \cdots \wedge A(v_k), \end{aligned}$$

which induces a unique linear map $\Lambda^k(A) \colon \Lambda^k(V) \to \Lambda^k(V')$. This is explicitly given by

$$\Lambda^k(A)(v_1 \wedge \cdots \wedge v_k) = A(v_1) \wedge \cdots \wedge A(v_k).$$

From this formula or the universal property one easily deduces the following:

Lemma 17.2.14.

- $\Lambda^k(BA) = \Lambda^k(B)\Lambda^k(A)$,
- $\Lambda^k(\mathrm{id}) = \mathrm{id}$.

17.2.3 The top exterior power and orientations

Let us take a closer look at the case $V = \mathbb{R}^k$. Then $\Lambda^k \mathbb{R}^k$ has a basis with a single element $e_1 \wedge \cdots \wedge e_k$, i.e. it is one-dimensional.

Example 17.2.15. For $k = 2$, $\mathbb{R}^2 \otimes \mathbb{R}^2$ is spanned by $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$ and $e_2 \otimes e_2$. In $\Lambda^2(\mathbb{R}^2)$ some additional antisymmetry rules are imposed. These for example say $e_1 \wedge e_2 = -e_2 \wedge e_1$. But they also say $e_1 \wedge e_1 = -e_1 \wedge e_1$ so $e_1 \wedge e_1 = 0$, and similarly $e_2 \wedge e_2 = 0$. Thus $\Lambda^2(\mathbb{R}^2)$ is indeed 1-dimensional spanned by $e_1 \wedge e_2$.

Thus for each linear map $A \colon \mathbb{R}^k \to \mathbb{R}^k$, the induced linear map $\Lambda^k(A) \colon \Lambda^k(\mathbb{R}^k) \to \Lambda^k(\mathbb{R}^k)$ is given by multiplication with a number, which for now we denote $d(A)$.

Example 17.2.16. For a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we can compute $d(A)$ by determining which multiple of $e_1 \wedge e_2$ the element $\Lambda^2(A)(e_1 \wedge e_2)$ is equal to. The latter is given by

$$\begin{aligned} A(e_1) \wedge A(e_2) &= (ae_1 + ce_2) \wedge (be_1 + de_2) \\ &= abe_1 \wedge e_1 + ade_1 \wedge e_2 + cbe_2 \wedge e_1 + cde_2 \wedge e_2 \\ &= (ad - bc)e_1 \wedge e_2. \end{aligned}$$

As the previous example shows, you are already familiar with the number $d(A)$.

Lemma 17.2.17. $d(A) = \det(A)$.

Sketch of proof. There are two ways to prove this.

You could use that the determinant is uniquely determined a small number of properties, namely that $\det(BA) = \det(B)\det(A)$ and its value on elementary matrices, upper-diagonal matrices, and permutation matrices. Indeed, using elementary matrices and permutation matrices you can row reduce all matrices to upper-diagonal ones. You then just need to verify that $d(BA) = d(B)d(A)$, which follows from $\Lambda^k(BA) = \Lambda^k(B)\Lambda^k(A)$, and that d takes the same value as det on elementary matrices, upper-diagonal matrices and permutation matrices.

Alternatively, you could just compute $A(e_1) \wedge \cdots \wedge A(e_k)$ directly. By linearity in each entry and observing that those terms where a basis vector is repeated are 0, you get

$$\begin{aligned} A(e_1) \wedge \cdots \wedge A(e_k) &= \sum_\sigma \left(\prod_{i=1}^k A_{i\sigma(i)} \right) e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)} \\ &= \sum_\sigma \left(\prod_{i=1}^k (-1)^{\epsilon(\sigma)} A_{i\sigma(i)} \right) e_1 \wedge \cdots \wedge e_k \\ &= \det(A) e_1 \wedge \cdots \wedge e_k . \square \end{aligned}$$

An invertible matrix $\det(A)$ is a composition of rotations and an upper-diagonal matrix with positive entries on the diagonal if and only if its determinant is positive. If the determinant is negative, then it is a composition of such matrices with a reflection in a hyperplane. If we think intuitively of an orientation has a notion of “handedness” (of “chirality” if you want a fancier term), then rotations and upper-diagonal matrices with positive entries on the diagonal should preserve this, but reflection should reverse this. This makes the following definition reasonable:

Definition 17.2.18. An *orientation* of a finite-dimensional $\mathbb{R}$-vector space V is a choice of a non-zero element of $\Lambda^{\dim(V)}(V)$ up to scaling by a *positive* real number.

This definition is set up so that an invertible linear map A preserves an orientation if and only if $\det(A) > 0$.

17.3 Orientations

17.3.1 Fiberwise constructions

We have already seen how natural constructions on vector spaces lead to natural construction on vector bundles, by repeating this construction fiberwise:

We proved that these constructions produce vector bundles by going to local trivializations, and then observing that the corresponding constructions on general linear maps are continuous or even smooth in the entries.

Let us repeat this with the top exterior power:

vector spaces	vector bundles
direct sum $V \oplus V'$	direct sum $E \oplus E'$
quotient V/V'	quotient E/E'
image $\mathrm{im}(A\colon V \to V')$	image $\mathrm{im}(G\colon E \to E')$ (if rank constant)
kernel $\ker(A\colon V \to V')$	kernel $\ker(G\colon E \to E')$ (if rank constant)

Definition 17.3.1. Let $p\colon E \to X$ be a vector bundle of dimension k. Then its *top exterior power* $\Lambda^k(p)\colon \Lambda^k(E) \to X$ is the vector bundle of dimension 1 given by $\bigsqcup_{x\in X} \Lambda^k(E_x)$. We topologize this as follows: for every local trivializations $\psi\colon p^{-1}(U) = \bigsqcup_{x\in U} E_x \to U \times \mathbb{R}^k$ we define declare that the local trivialization $(\Lambda^k(p))^{-1}(U) = \bigsqcup_{x\in U} \Lambda^k(E_x) \to U \times \Lambda^k(\mathbb{R}^k)$ given by taking $(x, v) \mapsto (x, \Lambda^k(\psi_x)(v))$ is a homeomorphism.

The transition functions of $\Lambda^k(E)$ are given by the determinant of the transition functions of E. Thus $\Lambda^k(E)$ will be a smooth vector bundle if E is a smooth vector bundle. Using this observation and similar ones for other exterior power or tensor products we can extend our table as follows:

vector spaces	vector bundles
top exterior power $\Lambda^{\dim(V)}(V)$	top exterior power $\Lambda^{\dim(E)}(E)$
tensor product $V \otimes V'$	tensor product $E \otimes E'$
exterior power $\Lambda^r(V)$	exterior power $\Lambda^r(E)$
symmetric power $\mathrm{Sym}^r(V)$	symmetric power $\mathrm{Sym}^r(E)$
dual V^*	dual E^*

17.3.2 Riemannian metrics

When thinking about smooth vector bundles it is sometimes helpful to have a Riemannian metric around:

Definition 17.3.2. A *Riemannian metric* is a section g of $(E \otimes E)^*$ such that on each fiber $g_x\colon E_x \otimes E_x \to \mathbb{R}$ is a positive definite symmetric bilinear form.

Lemma 17.3.3. *Every smooth vector bundle $p\colon E \to X$ admits a Riemannian metric, and this is unique up to homotopy.*

Proof. For each local trivialization $\psi\colon p^{-1}(U) \to U \times \mathbb{R}^k$ we can define on U the pullback along ψ^{-1} of the standard Riemann metric: for $v, v' \in E_x$,

$$(\psi^{-1})^* g_{\mathrm{std}}(v, v') := g_{\mathrm{std}}(\psi_x^{-1}(v), \psi_x^{-1}(v')).$$

Now take a partition of unity subordinate to an open cover of X by open subsets U of a local trivialization; $\eta_i\colon M \to [0, 1]$ supported in U_i. Then we define

$$g := \sum_i \eta_i \cdot (\psi_i^{-1})^* g_{\mathrm{std}}.$$

This is positive define and symmetric since these properties are preserved by convex linear combinations. For uniqueness, observe we can linearly interpolate between any two Riemannian metric. □

The main application of this is:

Lemma 17.3.4. *If $E' \subset E$ is a subbundle, then there is another subbundle $E'' \subset E$ such that $E' \oplus E'' \cong E$. This subbundle E'' is isomorphic to E/E'.*

Proof. Equip E with a Riemannian metric. Then we can take $E'' = (E')^\perp$, given by fibers $(E')^\perp_x := (E'_x)^\perp$. To get the second part, we observe that the map of vector bundles $E \to (E')^\perp$ given on fibers by orthogonal projection $E_x \to (E')^\perp_x$ with kernel given by E' and hence induces an isomorphism $E/E' \to (E')^\perp$. □

17.3.3 Orientations of vector bundles

Recall that a map which picks a single element of each fiber is called a section:

Definition 17.3.5. A *section* of a smooth vector bundle $p\colon E \to X$ is a smooth map $s\colon X \to E$ such that $p \circ s = \mathrm{id}_X$.

Example 17.3.6. Every smooth vector bundle has a 0-section $s_0\colon X \to E$ picking out the 0 in each fiber.

Example 17.3.7. A smooth section of TM is also known as a smooth vector field.

When we have a section $s\colon X \to E$ of a smooth vector bundle and a smooth function $g\colon X \to \mathbb{R}$, we can use fiberwise scalar multiplication to produce a new section $g \cdot s$.

Definition 17.3.8. An *orientation* of a smooth vector bundle $p\colon E \to B$ is an everywhere non-zero section s of $\Lambda^{\dim(E)}E$, up to the equivalence relation of scalar multiplication by an everywhere positive smooth function.

Thus, an orientation on E is smooth choice of non-zero elements of each $\Lambda^{\dim(E)}E_x$ up to scaling, that is, a smooth choice of orientation of each of vector spaces E_x.

Example 17.3.9. Trivial vector bundles always admit an orientation.

Example 17.3.10. A much more interesting example is the Moebius strip, i.e. the tautological bundle over $\mathbb{R}P^1$. We use the following straightforward observation: every section s of a smooth vector bundle $p\colon E \to B$ is homotopic to the 0-section. Indeed, take $H\colon B \times [0,1] \to E$ given by

$$(p,t) \longmapsto t \cdot s(p).$$

Using this we prove that the tautological bundle γ over $\mathbb{R}P^1$ (the one whose total space is the Moebius strip) does not admit an orientation. Let us identify $\mathbb{R}P^1$ with the 0-section. If this bundle did admit an orientation, there would be an everywhere non-zero section s and we would have $I_2(s, \mathbb{R}P^1) = 0$. But we also know that $I_2(s, \mathbb{R}P^1) = I_2(\mathbb{R}P^1, \mathbb{R}P^1)$, and latter is 1 by exhibiting a particular section transverse to the 0-section. This gives a contradiction.

A vector bundle E is said to be *orientable* if it admits an orientation.

Lemma 17.3.11. *A vector bundle E is orientable if $\Lambda^{\dim(E)}E$ is isomorphic to a trivial line bundle. Furthermore, an orientation is a trivialization of $\Lambda^{\dim(E)}E$ up to scalar multiplication by a smooth positive function.*

Proof. Indeed, a representative $s\colon X \to \Lambda^{\dim(E)}E$ of an orientation furnishes an isomorphism

$$\begin{aligned} X \times \mathbb{R} &\xrightarrow{\cong} \Lambda^{\dim(E)}E \\ (b,t) &\longmapsto t \cdot s(b). \end{aligned}$$

Conversely, an isomorphism $\phi\colon \Lambda^{\dim(E)} \cong X \times \mathbb{R}$ gives an everywhere non-vanishing section $s\colon X \to \Lambda^{\dim(E)}$ by $x \mapsto \phi^{-1}(x,1)$. □

If E is orientable, how many orientations does it admit? Given an orientation represented by s, any other orientation s' differs by scalar multiplication of s with an everywhere non-zero smooth function f. If we multiply f with an everywhere positive smooth function we get the same s', so the orientations are given by the set of everywhere non-zero smooth functions up to multiplication by everywhere positive smooth function. In other words, for each connected component of X we have to pick a choice of sign. We conclude that:

Lemma 17.3.12. *Let $\pi_0(X)$ denote the set of connected components of X, then if E is orientable the set of orientations is (non-canonically) given by the set of functions*

$$\pi_0(B) \longrightarrow \{\pm 1\}.$$

Given orientations for smooth vector bundles E, E' over X, you can produce a direct sum orientation on $E \oplus E'$. The observation you need is that there is a natural isomorphism

$$\begin{aligned} \Lambda^{\dim(E)}E \otimes \Lambda^{\dim(E')}E' &\xrightarrow{\cong} \Lambda^{\dim(E)+\dim(E')}(E \oplus E') \\ (v_1 \wedge \cdots \wedge v_{\dim(E)}) \otimes (v'_1 \wedge \cdots \wedge v'_{\dim(E')}) &\longmapsto v_1 \wedge \cdots \wedge v_{\dim(E)} \wedge v'_1 \wedge \cdots \wedge v'_{\dim(E')}. \end{aligned}$$

Thus trivializations of $\Lambda^{\dim(E)}E$ and $\Lambda^{\dim(E')}E'$ give a trivialization of $\Lambda^{\dim(E)}E \otimes \Lambda^{\dim(E')}E'$. Conversely, if $E = E' \oplus E''$ with E and E' oriented, the trivializations of E and E' give isomorphisms

$$B \times \mathbb{R} \cong \Lambda^{\dim(E')+\dim(E'')}(E' \oplus E'') \cong \Lambda^{\dim(E')}E' \otimes \Lambda^{\dim(E'')}E'' \cong \Lambda^{\dim(E'')}E'',$$

so an orientation of E''.

17.3.4 Orientations of manifolds

If M is a k-dimensional manifold, then TM is a k-dimensional smooth vector bundle M and hence $\Lambda^k TM$ is a 1-dimensional smooth vector bundle M, called the *orientation line bundle.*

Definition 17.3.13. An *orientation of M* is an orientation of TM.

Remark 17.3.14. An orientation of M is equivalent to a choice of "oriented" atlas inside its maximal atlas, where all transition functions are required to have total derivatives with positive determinant.

Let us give two examples of manifolds that are orientable and one which is not:

Example 17.3.15. If $M = S^1$, the tangent bundle is isomorphic to a trivial bundle and since $\Lambda^{\dim(E)}E = E$ for any 1-dimensional vector bundle so is its top exterior power. It hence admits exactly two orientations. These correspond to the clockwise and counterclockwise directions of the circle.

Example 17.3.16. If $M = *$, we have that $\Lambda^0 TM = \mathbb{R}$, so the point admits exactly two orientations. However, the one represented by $1 \in \mathbb{R}$ should obviously be our preferred choice.

Example 17.3.17. We claim that $\mathbb{R}P^2$ admits no orientation. If it did then so would $T\mathbb{R}P^2|_{\mathbb{R}P^1}$. This vector bundle is isomorphic to $T\mathbb{R}P^1 \oplus N\mathbb{R}P^1 \cong \mathbb{R} \oplus \gamma$, with γ the canonical bundle over $\mathbb{R}P^1$. This means its orientation line bundle is $\Lambda^2(\mathbb{R} \oplus \gamma) \cong \gamma$ and we proved above that γ does not admit an everywhere non-vanishing section, i.e. is not trivializable.

There are several constructions which produce new orientations on manifold form old ones:

Example 17.3.18. Given a manifold M with orientation, we can produce another orientation by multiplying a representative section $s\colon M \to \Lambda^k TM$ with -1. This is called *reversing the orientation* and we shall occasionally use the notion $-M$ for this.

Example 17.3.19. If M and N are manifolds with orientations, then we get a direct sum orientation on $M \times N$, as $T_{(p,p')}(M \times N) \cong T_pM \oplus T_{p'}N$.

To phrase this in terms of vector bundles, we need a generalization of the restriction of vector bundles: given any map $f\colon X' \to X$ we can *pull back* a vector bundle $p\colon E \to X$ to X' by setting $f^*E = \bigsqcup_{x' \in X} E_{f(x')}$. In the language of vector bundles we have $T(M \times N) \cong \pi_1^* TM \oplus \pi_2^* TN$.

Example 17.3.20. If $Z \subset N$ is a submanifold and both N and Z are oriented, then the isomorphism $TN|_Z \cong NZ \oplus TZ$ shows that NZ also comes with an orientation.

Example 17.3.21. Suppose we have a smooth map $f\colon M \to N$ with M and N oriented, and $Z \subset M$ an oriented submanifold such that $f \pitchfork Z$. Then $f^{-1}(Z)$ is a submanifold and its tangent bundle satisfies $f^*NZ \oplus Tf^{-1}(Z) \cong TM|_{f^{-1}(Z)}$. Since both $TM|_{f^{-1}(Z)}$ and f^*NZ comes with orientations, we get an orientation of $Tf^{-1}(Z)$.

17.3.5 Induced orientation on the boundary

If M is a manifold with boundary ∂M, then its boundary ∂M inherits an orientation, canonically so once we fix a single convention. To do so, it is convenient to pick a Riemannian metric on M, that is, on TM. Then the restriction $TM|_{\partial M}$ inherits a Riemannian metric and thus splits as $T\partial M \oplus (T\partial M)^{\perp}$, the latter being a line bundle.

From our discussion of collars, we know that there exist a smooth function $\chi\colon M \to [0,\infty)$ such that $\chi^{-1}(0) = \partial M$ and for each $p \in \partial M$, $d_p\chi$ is non-vanishing on some vector $v \in T_pM \backslash T_p\partial M$. This vector v decomposes as a sum of a vector $v_\partial \in T_p\partial M$ and a vector $v_\perp \in (T_p\partial M)^\perp$. Since χ is constant on ∂M, v_∂ is zero so $v_\perp$ is non-zero. Hence the restriction $d_p\chi\colon (T_p\partial M)^\perp \to \mathbb{R}$ is non-zero.

We call a vector $v \in (T_p\partial M)^\perp$ such that $d_p\chi(v) < 0$ *outward pointing.* Such a vector is unique up to scaling by a positive real number. In particular, there is a canonical section n of $(TM|_{\partial M})^\perp$ given at $p \in \partial M$ by the unique element n_p of $(T_p\partial M)^\perp$ such that $d_p\chi(n_p) = 1$.

Every vector $v \in V$ provides a linear map $v \wedge -\colon \Lambda^{k-1}(V) \to \Lambda^k(V)$. This generalizes to a map of vector bundles

$$\begin{aligned} \Lambda^{k-1}(T\partial M) &\longrightarrow \Lambda^k(TM|_{\partial M}) \\ w &\longmapsto n \wedge w \end{aligned}$$

of vector bundles, by thinking of $\Lambda^{k-1}(T\partial M)$ as a linear subspace of $\Lambda^{k-1}(TM|_{\partial M})$ using the inclusion of $T\partial M$ into $TM|_{\partial M}$.

Lemma 17.3.22. *If an orientation of M is represented by the section s of Λ^kTM, then there is a unique orientation of M which is represented by a section $\bar{s}$ of $\Lambda^{k-1}T\partial M$ satisfying $n \wedge \bar{s} = s$.*

Proof. For each $p \in \partial M$, fix a basis $e_1, \ldots, e_{k-1}$ of $T_p\partial M$. By adding n_p we get a basis of T_pM. Then $\bar{s}(p)$ is by definition $\bar{c}(p) \cdot e_1 \wedge \cdots \wedge e_{k-1}$ for some $\bar{c} \in \mathbb{R}$, and $s(p)$ similarly is $c(p) \cdot n_p \wedge e_1 \wedge \cdots \wedge e_{k-1}$ for some $c(p) \in \mathbb{R}$. From the equation

$$n_p \wedge (\bar{c}(p) \cdot e_1 \wedge \cdots \wedge e_{k-1}) = c(p) \cdot n_p \wedge e_1 \wedge \cdots \wedge e_{k-1}$$

we read off $\bar{c}(p) = c(p)$, so $\bar{s}$ is uniquely determined by n and s.

Firstly $\bar{s}$, up to multiplication by a positive smooth function, is independent of the choice of representative s: if s changes by multiplying it with positive smooth function, so does $\bar{s}$.

Next, we have to verify the orientation is independent of the choice of Riemannian metric g and smooth function χ. Modifying the latter just changes n by scalar multiplication by a positive smooth function, and hence has the same effect on $\bar{s}$. If we vary g, then n_p gets replaced by $n'_p = an_p + \sum_{i=1}^{k-1} a_ie_i$ with $a > 0$ so

$$n'_p \wedge (\bar{c}(p) \cdot e_1 \wedge \cdots \wedge e_{k-1}) = a \cdot n_p \wedge (\bar{c}(p) \cdot e_1 \wedge \cdots \wedge e_{k-1}),$$

and again $\bar{s}$ just changes by scalar multiplication by a positive smooth function. □

Definition 17.3.23. If M is oriented, we shall consider ∂M as oriented by the orientation produced in the previous lemma. We refer to this as the *induced orientation.*

Example 17.3.24. There is a preferred choice of orientation on $[0,1]$, namely using $1 \in \Lambda^1 T_p[0,1] \cong T_p[0,1] \cong \mathbb{R}$. Then

$$\partial[0,1] \cong \{1\} - \{0\},$$

where, for an oriented manifold N, $-N$ denotes the same manifold with opposite orientation.

More generally, if M is oriented without boundary, then

$$\partial([0,1] \times M) = M \times \{1\} - M \times \{0\}.$$

However, if we do $\partial(M \times [0,1])$ we get $(-1)^{\dim(M)}(M \times \{1\} - M \times \{0\})$. This is an unfortunate clash of our conventions for orientations and notation for homotopies.

Example 17.3.25. Generalizing Example 18.3.21 to the case that M has boundary and $f \pitchfork Z$, $\partial f \pitchfork Z$ we get that $\partial f^{-1}(Z) = (\partial f)^{-1}(Z)$ comes with two orientations: one as the boundary of an oriented manifold and one as the inverse image of an oriented manifold. These are not equal but satisfy

$$\partial f^{-1}(Z) = (-1)^{\operatorname{codim}(Z)}(\partial f)^{-1}(Z).$$

17.4 Integral intersection theory

Chapter 3 of [GP10] upgrades the mod 2 intersection theory to an integral version. The main input is the observation that

$$\partial[0,1] \cong \{1\} - \{0\}$$

and the classification of compact 1-dimensional manifolds lead to the following result:

Proposition 17.4.1. *If M is a compact oriented 1-dimensional manifold, then the number of positively-oriented points in ∂M is equal to the number of negatively-oriented points.*

So we can define intersection numbers with values in $\mathbb{Z}$ instead of $\mathbb{Z}/2$:

Definition 17.4.2. Suppose that Y is a compact oriented manifold without boundary, M is an oriented manifold and $Z \subset M$ is an oriented submanifold such that $\dim(Y) + \dim(Z) = \dim(M)$.

Let $f_0 \colon Y \to M$ be a smooth map. Then the *intersection number* $I(f_0, Z)$ is defined as follows: take f_1 homotopic to f_0 and transverse to Z, and set

$$I(f_0, Z) = \sum_{p \in f_1^{-1}(Z)} \textit{orientation of } p.$$

One proceeds as before, using Proposition 18.4.1 in place of the fact that the number of points in the boundary of a compact 1-dimensional manifold is even, to prove that $I(f_0, Z)$ is well-defined and establish its basic properties. You can then easily define integral versions of the degree of a map and the winding numbers, and use these to great effect.

Example 17.4.3. With these definitions in hand, the mod 2 linking numbers of Section **??** generalize to integer linking numbers.

17.5 Problems

Problem 43 (Codimension 1 submanifolds are orientable)**.** Use the Jordan–Brouwer separation theorem to prove that if $M \subset \mathbb{R}^k$ is a compact codimension 1 submanifold, then it is orientable.

Problem 44. Define a *degree* $\deg(f) \in \mathbb{Z}$ of a smooth map $f\colon M \to N$ between compact oriented smooth manifolds of the same dimension, which reduces to $\deg_2(f)$ modulo 2.

Problem 45. Use partitions of unity to prove that any vector $v \in T_pM$ is the value at x of some smooth vector field X on M.

Chapter 18

Orientations and integral intersection theory

The next part of these lectures will be devoted defining de Rham cohomology, developing computational tools for it, and drawing interesting topological conclusions from it. A prerequisite for some of this material will be the notion of an orientation. We define this today, and give a taste of integral intersection theory, which we will not cover in detail in the course.

Convention 18.0.1. All vector spaces are finite-dimensional and over $\mathbb{R}$ unless mentioned otherwise.

18.1 What is an orientation on a manifold?

We start with an intuitive description of orientations, before giving rigorous definitions:

> an orientation of a manifold is "a smooth family of orientations of each of the tangent spaces T_pM."

An *orientation* on a vector space such as T_pM specifies for each of its ordered bases whether it is "positively oriented" or "negatively oriented," with the following requirement: if one ordered basis can be obtained from another by applying an invertible matrix A to each of its vectors, then we say they are *similarly oriented* if and only if $\det(A) > 0$. Since $\mathrm{GL}_n(\mathbb{R})$ has two path components, this is equivalent to saying homotopic bases are similarly oriented and reflecting a single basis vector changes the orientation of the basis. That an orientation depends smoothly on $p \in M$ means that if you move a positively oriented basis around M, it stays positive (and of course the same is true for negatively oriented bases).

Example 18.1.1. For the circle S^1, an orientation is a choice of "positive direction" along the circle. There are two such choices: counterclockwise and clockwise.

Example 18.1.2. The real projective plane $\mathbb{R}P^2$ admits no orientation. Suppose it did, then starting with a basis e_1, e_2 at some point, say positively oriented, we can move it around $\mathbb{R}P^2$ and return to $e_1, -e_2$. This must simultaneously be positively oriented (since moving a basis around should not change how it is oriented) and negatively oriented (since it is obtained from a positively oriented by reflecting a basis vector). This gives a contradiction.

Example 18.1.3. An LCD display is made from a *nematic crystal*, consisting of long thin filaments. These prefer to be aligned the same way, so locally such a crystal has a *order parameter* given by a direction in $\mathbb{R}^3$. This is an element of $\mathbb{R}P^2$, a non-orientable manifold. [1]

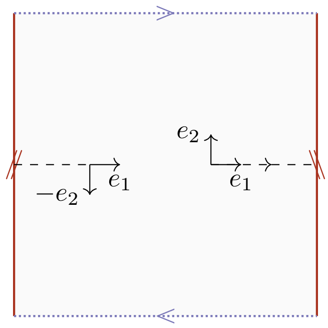


Figure 18.1 Moving a basis around $\mathbb{R}P^2$ can return it with opposite orientation.

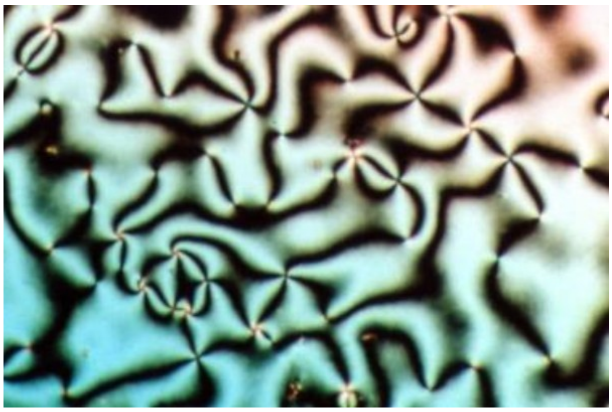


Figure 18.2 An nematic crystal (from https://en.wikipedia.org/wiki/Liquid_crystal).

You can find more examples in the following table:

18.2 A recollection of multilinear algebra

Linear algebra concerns not only the study of vector spaces and linear maps between them, but also of multilinear maps with various properties. This is closely related to the study of tensor products and variations thereof.

18.2.1 Tensor products

Definition 18.2.1. A *bilinear map* is a function $b\colon V \times V' \to W$ which is linear in each variable.

[1] See http://www.lassp.cornell.edu/sethna/pubPDF/OrderParameters.pdf.

orientable	not orientable
spheres S^n	real projective spaces $\mathbb{R}P^{2n}$ $(n \geqslant 1)$
surfaces of genus $g \geqslant 1$	Klein bottle
Lie groups	
Lens spaces	
Poincaré homology sphere	
Complex projective spaces	
Quaternionic projective spaces	
$K3$ surface	
Whitehead manifold	

Definition 18.2.2. The *tensor product* $V \otimes W$ is the quotient of the free $\mathbb{R}$-vector space on the set $V \times V'$, whose basis elements we shall denote (v, v'), by the subspace spanned by the elements

$$((v_1 + v_2), v') - (v_1, v') - (v_2, v'),$$

$$(v, (v_1' + v_2')) - (v, v_1') - (v, v_2'),$$

$$(av, v') - a(v, v'),$$

$$(v, av') - a(v, v').$$

We will denote the equivalence class of (v, v') by $v \otimes v'$.

Example 18.2.3. The tensor product $\mathbb{R}^k \otimes \mathbb{R}^{k'}$ has a basis given by $e_i \otimes e_j'$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k'$.

The relations are designed to make

$$\begin{aligned} b_0 \colon V \times W &\longrightarrow V \otimes V' \\ (v, v') &\longmapsto v \otimes v' \end{aligned}$$

bilinear. It is in fact the initial bilinear map:

Lemma 18.2.4. *For every bilinear map $b \colon V \times V' \to W$ there is a unique linear map $\beta \colon V \otimes V' \to W$ such that $b = \beta \circ b_0$.*

Proof. There is a unique linear map $\mathbb{R}[V \times V'] \to W$ given by $(v, v') \mapsto b(v, v')$. Since b is bilinear this factors over $V \otimes V'$, determining a linear map $\beta \colon V \otimes V' \to W$ satisfying $b(v, v') = \beta(v \otimes v') = \beta(b_0(v, v'))$. Since $V \otimes V'$ is generated by the elements $b_0(v, v')$, this determines β uniquely. $\square$

Remark 18.2.5. This universal property satisfied by the tensor product determines it uniquely up to linear isomorphism.

There is a similar one-to-one correspondence of multilinear maps $V_1 \times \cdots \times V_k \to W$ with linear map $V_1 \otimes \cdots \otimes V_k \to W$.

Example 18.2.6. The universal property tells us what tensor product of a single or no vector spaces is. A multilinear map $V \to W$ is just a linear map, so a tensor product of a single vector space V is just V again.

The empty product of sets is a point, because such a product receives a unique map from every other set. A multilinear map from an empty product is hence a map from a point to V, with no condition imposed, so just an element of V. This is the same as a linear map $\mathbb{R} \to V$. Hence an empty tensor product is $\mathbb{R}$ itself.

18.2.2 Alternating multilinear maps

When all vector spaces V_i in the domain of a multilinear map are the same V, we can require additional symmetry properties. Of specific interest to us are the alternating multilinear maps, though the story for symmetric multilinear maps is similar:

Definition 18.2.7. An *alternating multilinear map* is a multilinear map $w\colon V^k \to W$ which satisfies $w(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (-1)^{\epsilon(\sigma)} w(v_1, \ldots, v_k)$ for all $v_1, \ldots, v_k \in V$ and permutations σ of $\{1, \ldots, k\}$. Here $\epsilon(\sigma) \in \mathbb{Z}/2$ is the sign of the permutation.

Example 18.2.8. The sign of a permutation is uniquely determined by demanding it is a homomorphism and it sends a transposition to the unique non-identity element of $\mathbb{Z}/2$.

There is also an initial alternating multilinear map.

Definition 18.2.9. The *kth exterior power* $\Lambda^k V$ is the quotient of $V^{\otimes k}$ by the subspace spanned by the elements

$$v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} - (-1)^{\epsilon(\sigma)} v_1 \otimes \ldots \otimes v_k \text{ with } \sigma \in \Sigma_k.$$

We will denote the image of $v_1 \otimes \cdots \otimes v_k$ by $v_1 \wedge \cdots \wedge v_k$.

Example 18.2.10. $\Lambda^2 \mathbb{R}^n$ has a basis $e_i \wedge e_j$ for $1 \leqslant i < j \leqslant n$. It is a well-known mistake to think that every element of an exterior product is of the form $v_1 \wedge v_2$. This is not the case, e.g. $e_1 \wedge e_2 + e_3 \wedge e_4$ can't be written this way.

Example 18.2.11. $\Lambda^0 V$ is the quotient of $(V)^{\otimes 0} = \mathbb{R}$ by the trivial subspace, so is equal to $\mathbb{R}$.

The subspace in Definition 18.2.9 is designed to make

$$\begin{aligned} w_0\colon V^k &\longrightarrow \Lambda^k V \\ (v_1, \ldots, v_k) &\longmapsto v_1 \wedge \cdots \wedge v_k \end{aligned}$$

alternating multilinear. This satisfies:

Lemma 18.2.12. *For every alternating multilinear map $w\colon V^k \to W$ there is a unique linear map $\omega\colon \Lambda^k V \to W$ such that $w = \omega \circ w_0$.*

Remark 18.2.13. This universal property tells us that there is a map $V^{\otimes k} \to \Lambda^k V$ corresponding to a natural assignment of an alternating multilinear map $w(b)\colon V^k \to W$ to each multilinear map $b\colon V^k \to W$. It is given by anti-symmetrizing:

$$w(b)(v_1, \dots, v_k) = \sum_{\sigma \in \Sigma_k} b(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

The construction of $\Lambda^k V$ is natural in V: whenever we have a linear map $A\colon V \to V'$, there is an alternating multilinear map

$$\begin{aligned} V^{\times k} &\longrightarrow \Lambda^k V' \\ (v_1, \dots, v_k) &\longmapsto A(v_1) \wedge \cdots \wedge A(v_k), \end{aligned}$$

which induces a unique linear map $\Lambda^k A\colon \Lambda^k V \to \Lambda^k V'$. This is explicitly given by

$$(\Lambda^k A)(v_1 \wedge \cdots \wedge v_k) = A(v_1) \wedge \cdots \wedge A(v_k).$$

From this formula or the universal property one easily deduces the following:

Lemma 18.2.14.

- $\Lambda^k(BA) = (\Lambda^k B)(\Lambda^k A)$,
- $\Lambda^k \mathrm{id} = \mathrm{id}$.

18.2.3 The top exterior power and orientations

Let us take a closer look at the case $V = \mathbb{R}^k$. Then $\Lambda^k \mathbb{R}^k$ has a basis given by the single element $e_1 \wedge \cdots \wedge e_k$, so in particular is one-dimensional.

Example 18.2.15. For $k = 2$, $\mathbb{R}^2 \otimes \mathbb{R}^2$ is spanned by $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$ and $e_2 \otimes e_2$. In $\Lambda^2(\mathbb{R}^2)$ some additional antisymmetry rules are imposed. These for example say $e_1 \wedge e_2 = -e_2 \wedge e_1$. But they also say $e_1 \wedge e_1 = -e_1 \wedge e_1$ so $e_1 \wedge e_1 = 0$, and similarly $e_2 \wedge e_2 = 0$. Thus $\Lambda^2(\mathbb{R}^2)$ is indeed 1-dimensional spanned by $e_1 \wedge e_2$.

Thus for each linear map $A\colon \mathbb{R}^k \to \mathbb{R}^k$, the induced linear map $\Lambda^k(A)\colon \Lambda^k(\mathbb{R}^k) \to \Lambda^k(\mathbb{R}^k)$ is given by multiplication with a number, which for now we denote $d(A)$.

Example 18.2.16. For a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we can compute $d(A)$ by determining which multiple of $e_1 \wedge e_2$ the element $\Lambda^2(A)(e_1 \wedge e_2)$ is equal to. The latter is given by

$$\begin{aligned} A(e_1) \wedge A(e_2) &= (ae_1 + ce_2) \wedge (be_1 + de_2) \\ &= abe_1 \wedge e_1 + ade_1 \wedge e_2 + cbe_2 \wedge e_1 + cde_2 \wedge e_2 \\ &= (ad - bc)e_1 \wedge e_2. \end{aligned}$$

As the previous example shows, you are already familiar with the number $d(A)$.

Lemma 18.2.17. $d(A) = \det(A)$.

Sketch of proof. There are two ways to prove this.

You could use that the determinant is uniquely determined a small number of properties, namely that $\det(BA) = \det(B)\det(A)$ and its value on elementary matrices, upper-diagonal matrices, and permutation matrices. Indeed, using elementary matrices and permutation matrices you can row reduce all matrices to upper-diagonal ones. You then just need to verify that $d(BA) = d(B)d(A)$, which follows from $\Lambda^k(BA) = \Lambda^k(B)\Lambda^k(A)$, and that d takes the same value as det on elementary matrices, upper-diagonal matrices and permutation matrices.

Alternatively, you could just compute $A(e_1) \wedge \cdots \wedge A(e_k)$ directly. By linearity in each entry and observing that those terms where a basis vector is repeated are 0, you get

$$\begin{aligned} A(e_1) \wedge \cdots \wedge A(e_k) &= \sum_\sigma \left(\prod_{i=1}^k A_{i\sigma(i)} \right) e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)} \\ &= \sum_\sigma \left(\prod_{i=1}^k (-1)^{\epsilon(\sigma)} A_{i\sigma(i)} \right) e_1 \wedge \cdots \wedge e_k \\ &= \det(A) e_1 \wedge \cdots \wedge e_k. \square \end{aligned}$$

An invertible matrix $\det(A)$ is a composition of rotations and an upper-diagonal matrix with positive entries on the diagonal if and only if its determinant is positive. If the determinant is negative, then it is a composition of such a matrix with a reflection in a hyperplane. If we think intuitively of an orientation has a notion of "handedness" (of "chirality" if you want a fancier term), then rotations and upper-diagonal matrices with positive entries on the diagonal should preserve this, but reflection should reverse this. This makes the following definition reasonable:

Definition 18.2.18. An *orientation* of a finite-dimensional $\mathbb{R}$-vector space V is a choice of a non-zero element of $\Lambda^{\dim(V)} V$ up to scaling by a *positive* real number.

This definition is set up so that an invertible linear map A preserves an orientation if and only if $\det(A) > 0$.

18.3 Orientations

18.3.1 Fiberwise constructions

We have already seen how natural constructions on vector spaces lead to natural construction on vector bundles, by repeating this construction fiberwise:

We proved that these constructions produce vector bundles by going to local trivializations, and then observing that the corresponding constructions on general linear maps are continuous or even smooth in the entries. Let us repeat this with the top exterior power:

vector spaces	vector bundles
direct sum $V \oplus V'$	direct sum $E \oplus E'$
quotient V/V'	quotient E/E'
image $\mathrm{im}(A\colon V \to V')$	image $\mathrm{im}(G\colon E \to E')$ (if rank constant)
kernel $\ker(A\colon V \to V')$	kernel $\ker(G\colon E \to E')$ (if rank constant)

Definition 18.3.1. Let $p\colon E \to X$ be a vector bundle of dimension k. Then its *top exterior power* $\Lambda^k(p)\colon \Lambda^k(E) \to X$ is the vector bundle of dimension 1 given by $\bigsqcup_{x \in X} \Lambda^k(E_x)$. We topologise this as follows: for every local trivializations $\psi\colon p^{-1}(U) = \bigsqcup_{x \in U} E_x \to U \times \mathbb{R}^k$ we define declare that the local trivialization $(\Lambda^k(p))^{-1}(U) = \bigsqcup_{x \in U} \Lambda^k(E_x) \to U \times \Lambda^k(\mathbb{R}^k)$ given by taking $(x, v) \mapsto (x, \Lambda^k(\psi_x)(v))$ is a homeomorphism.

The transition functions of $\Lambda^k(E)$ are given by the determinant of the transition functions of E. Thus $\Lambda^k(E)$ will be a smooth vector bundle if E is a smooth vector bundle. Using this observation and similar ones for other exterior power or tensor products we can extend our table as follows:

vector spaces	vector bundles
top exterior power $\Lambda^{\dim(V)}(V)$	top exterior power $\Lambda^{\dim(E)}(E)$
tensor product $V \otimes V'$	tensor product $E \otimes E'$
exterior power $\Lambda^r(V)$	exterior power $\Lambda^r(E)$
symmetric power $\mathrm{Sym}^r(V)$	symmetric power $\mathrm{Sym}^r(E)$
dual V^*	dual E^*

18.3.2 Riemannian metrics

When thinking about smooth vector bundles it is sometimes helpful to have a Riemannian metric around:

Definition 18.3.2. A *Riemannian metric* is a section g of $(E \otimes E)^*$ such that on each fiber $g_x\colon E_x \otimes E_x \to \mathbb{R}$ is a positive definite symmetric bilinear form.

Lemma 18.3.3. *Every smooth vector bundle $p\colon E \to X$ admits a Riemannian metric, and this is unique up to homotopy.*

Proof. For each local trivialization $\psi\colon p^{-1}(U) \to U \times \mathbb{R}^k$ we can define on U the pullback along ψ^{-1} of the standard Riemann metric: for $v, v' \in E_x$,

$$(\psi^{-1})^* g_{\mathrm{std}}(v, v') := g_{\mathrm{std}}(\psi_x^{-1}(v), \psi_x^{-1}(v')).$$

Now take a partition of unity subordinate to an open cover of X by open subsets U of a local trivialization; $\eta_i\colon M \to [0, 1]$ supported in U_i. Then we define

$$g := \sum_i \eta_i \cdot (\psi_i^{-1})^* g_{\mathrm{std}}.$$

This is positive define and symmetric since these properties are preserved by convex linear combinations. For uniqueness, observe we can linearly interpolate between any two Riemannian metric. □

The main application of this is:

Lemma 18.3.4. *If $E' \subset E$ is a subbundle, then there is another subbundle $E'' \subset E$ such that $E' \oplus E'' \cong E$. This subbundle E'' is isomorphic to E/E'.*

Proof. Equip E with a Riemannian metric. Then we can take $E'' = (E')^\perp$, given by fibres $(E')^\perp_x := (E'_x)^\perp$. To get the second part, we observe that the map of vector bundles $E \to (E')^\perp$ given on fibres by orthogonal projection $E_x \to (E')^\perp_x$ with kernel given by E' and hence induces an isomorphism $E/E' \to (E')^\perp$. □

18.3.3 Orientations of vector bundles

Recall that a map which picks a single element of each fibre is called a section:

Definition 18.3.5. A *section* of a smooth vector bundle $p\colon E \to X$ is a smooth map $s\colon X \to E$ such that $p \circ s = \mathrm{id}_X$.

Example 18.3.6. Every smooth vector bundle has a 0-section $s_0\colon X \to E$ picking out the 0 in each fibre.

Example 18.3.7. A smooth section of TM is also known as a smooth vector field.

When we have a section $s\colon X \to E$ of a smooth vector bundle and a smooth function $g\colon X \to \mathbb{R}$, we can use fiberwise scalar multiplication to produce a new section $g \cdot s$.

Definition 18.3.8. An *orientation* of a smooth vector bundle $p\colon E \to B$ is an everywhere non-zero section s of $\Lambda^{\dim(E)}E$, up to the equivalence relation of scalar multiplication by an everywhere positive smooth function.

Thus, an orientation on E is smooth choice of non-zero elements of each $\Lambda^{\dim(E)}E_x$ up to scaling, that is, a smooth choice of orientation of each of vector spaces E_x.

Example 18.3.9. Trivial vector bundles always admit an orientation.

Example 18.3.10. A more interesting example is the Moebius strip, i.e. the tautological bundle over $\mathbb{R}P^1$. We use the following straightforward observation: every section s of a smooth vector bundle $p\colon E \to B$ is homotopic to the 0-section. Indeed, take $H\colon B \times [0,1] \to E$ given by

$$(p,t) \longmapsto t \cdot s(p).$$

Using this we prove that the tautological bundle γ over $\mathbb{R}P^1$ (the one whose total space is the Moebius strip) does not admit an orientation. Let us identify $\mathbb{R}P^1$ with the 0-section. If this bundle did admit an orientation, there would be an everywhere non-zero section s and we would have $I_2(s, \mathbb{R}P^1) = 0$. But we also know that $I_2(s, \mathbb{R}P^1) = I_2(\mathbb{R}P^1, \mathbb{R}P^1)$, and latter is 1 by exhibiting a particular section transverse to the 0-section. This gives a contradiction.

A vector bundle E is said to be *orientable* if it admits an orientation.

Lemma 18.3.11. *A vector bundle E is orientable if $\Lambda^{\dim(E)}E$ is isomorphic to a trivial line bundle. Furthermore, an orientation is a trivialization of $\Lambda^{\dim(E)}E$ up to scalar multiplication by a smooth positive function.*

Proof. Indeed, a representative $s\colon X \to \Lambda^{\dim(E)}E$ of an orientation furnishes an isomorphism

$$\begin{aligned} X \times \mathbb{R} &\xrightarrow{\cong} \Lambda^{\dim(E)}E \\ (b,t) &\longmapsto t \cdot s(b). \end{aligned}$$

Conversely, an isomorphism $\phi\colon \Lambda^{\dim(E)} \cong X \times \mathbb{R}$ gives an everywhere non-vanishing section $s\colon X \to \Lambda^{\dim(E)}$ by $x \mapsto \phi^{-1}(x,1)$. □

If E is orientable, how many orientations does it admit? Given an orientation represented by s, any other orientation s' differs by scalar multiplication of s with an everywhere non-zero smooth function f. If we multiply f with an everywhere positive smooth function we get the same s', so the orientations are given by the set of everywhere non-zero smooth functions up to multiplication by everywhere positive smooth function. In other words, for each connected component of X we have to pick a choice of sign. We conclude that:

Lemma 18.3.12. *Let $\pi_0(X)$ denote the set of connected components of X, then if E is orientable the set of orientations is (non-canonically) given by the set of functions*

$$\pi_0(B) \longrightarrow \{\pm 1\}.$$

Given orientations for smooth vector bundles E, E' over X, you can produce a direct sum orientation on $E \oplus E'$. The observation you need is that there is a natural isomorphism

$$\begin{aligned} \Lambda^{\dim(E)}E \otimes \Lambda^{\dim(E')}E' &\xrightarrow{\cong} \Lambda^{\dim(E)+\dim(E')}(E \oplus E') \\ (v_1 \wedge \cdots \wedge v_{\dim(E)}) \otimes (v_1' \wedge \cdots \wedge v'_{\dim(E')}) &\longmapsto v_1 \wedge \cdots \wedge v_{\dim(E)} \wedge v_1' \wedge \cdots \wedge v'_{\dim(E')}. \end{aligned}$$

Thus trivializations of $\Lambda^{\dim(E)}E$ and $\Lambda^{\dim(E')}E'$ give a trivialization of $\Lambda^{\dim(E)}E \otimes \Lambda^{\dim(E')}E'$.

Conversely, if $E = E' \oplus E''$ with E and E' oriented, the trivializations of E and E' give isomorphisms

$$B \times \mathbb{R} \cong \Lambda^{\dim(E')+\dim(E'')}(E' \oplus E'') \cong \Lambda^{\dim(E')}E' \otimes \Lambda^{\dim(E'')}E'' \cong \Lambda^{\dim(E'')}E'',$$

so an orientation of E''.

18.3.4 Orientations of manifolds

If M is a k-dimensional manifold, then TM is a k-dimensional smooth vector bundle M and hence $\Lambda^k TM$ is a 1-dimensional smooth vector bundle M, called the *orientation line bundle*.

Definition 18.3.13. An *orientation of* M is an orientation of TM.

Remark 18.3.14. An orientation of M is equivalent to a choice of "oriented" atlas inside its maximal atlas, where all transition functions are required to have total derivatives with positive determinant.

Let us give two examples of manifolds that are orientable and one which is not:

Example 18.3.15. If $M = S^1$, the tangent bundle is isomorphic to a trivial bundle and since $\Lambda^{\dim(E)}E = E$ for any 1-dimensional vector bundle so is its top exterior power. It hence admits exactly two orientations. These correspond to the clockwise and counterclockwise directions of the circle.

Example 18.3.16. If $M = *$, we have that $\Lambda^0 TM = \mathbb{R}$, so the point admits exactly two orientations. However, the one represented by $1 \in \mathbb{R}$ should obviously be our preferred choice.

Example 18.3.17. We claim that $\mathbb{R}P^2$ admits no orientation. If it did then so would $T\mathbb{R}P^2|_{\mathbb{R}P^1}$. This vector bundle is isomorphic to $T\mathbb{R}P^1 \oplus N\mathbb{R}P^1 \cong \mathbb{R} \oplus \gamma$, with γ the canonical bundle over $\mathbb{R}P^1$. This means its orientation line bundle is $\Lambda^2(\mathbb{R} \oplus \gamma) \cong \gamma$ and we proved above that γ does not admit an everywhere non-vanishing section, i.e. is not trivializable.

There are several constructions which produce new orientations on manifold form old ones:

Example 18.3.18. Given a manifold M with orientation, we can produce another orientation by multiplying a representative section $s\colon M \to \Lambda^k TM$ with -1. This is called *reversing the orientation* and we shall occasionally use the notion $-M$ for this.

Example 18.3.19. If M and N are manifolds with orientations, then we get a direct sum orientation on $M \times N$, as $T_{(p,p')}(M \times N) \cong T_pM \oplus T_{p'}N$.

To phrase this in terms of vector bundles, we need a generalization of the restriction of vector bundles: given any map $f\colon X' \to X$ we can *pull back* a vector bundle $p\colon E \to X$ to X' by setting $f^*E = \bigsqcup_{x' \in X} E_{f(x')}$. In the language of vector bundles we have $T(M \times N) \cong \pi_1^* TM \oplus \pi_2^* TN$.

Example 18.3.20. If $Z \subset N$ is a submanifold and both N and Z are oriented, then the isomorphism $TN|_Z \cong NZ \oplus TZ$ shows that NZ also comes with an orientation.

Example 18.3.21. Suppose we have a smooth map $f\colon M \to N$ with M and N oriented, and $Z \subset M$ an oriented submanifold such that $f \pitchfork Z$. Then $f^{-1}(Z)$ is a submanifold and its tangent bundle satisfies $f^*NZ \oplus Tf^{-1}(Z) \cong TM|_{f^{-1}(Z)}$. Since both $TM|_{f^{-1}(Z)}$ and f^*NZ comes with orientations, we get an orientation of $Tf^{-1}(Z)$.

18.3.5 Induced orientation on the boundary

If M is a manifold with boundary ∂M, then its boundary ∂M inherits an orientation, canonically so once we fix a single convention. To do so, it is convenient to pick a Riemannian metric on M, that is, on TM. Then the restriction $TM|_{\partial M}$ inherits a Riemannian metric and thus splits as $T\partial M \oplus (T\partial M)^\perp$, the latter being a line bundle.

By Lemma 14.2.7, there exist a smooth function $\chi\colon M \to [0,\infty)$ such that $\chi^{-1}(0) = \partial M$ and for each $p \in \partial M$, $d_p\chi$ is non-vanishing on some vector $v \in T_pM \backslash T_p\partial M$. This vector v decomposes as a sum of a vector $v_\partial \in T_p\partial M$ and a vector $v_\perp \in (T_p\partial M)^\perp$. Since χ is constant on ∂M, v_∂ is zero so $v_\perp$ is non-zero. Hence the restriction $d_p\chi\colon (T_p\partial M)^\perp \to \mathbb{R}$ is non-zero.

We call a vector $v \in (T_p\partial M)^\perp$ such that $d_p\chi(v) < 0$ *outward pointing.* Such a vector is unique up to scaling by a positive real number. In particular, there is a canonical section n of $(TM|_{\partial M})^\perp$ given at $p \in \partial M$ by the unique element n_p of $(T_p\partial M)^\perp$ such that $d_p\chi(n_p) = 1$.

Every vector $v \in V$ provides a linear map $v \wedge -\colon \Lambda^{k-1}(V) \to \Lambda^k(V)$. This generalizes to a map of vector bundles

$$\begin{aligned} \Lambda^{k-1}(T\partial M) &\longrightarrow \Lambda^k(TM|_{\partial M}) \\ w &\longmapsto n \wedge w \end{aligned}$$

of vector bundles, by thinking of $\Lambda^{k-1}(T\partial M)$ as a linear subspace of $\Lambda^{k-1}(TM|_{\partial M})$ using the inclusion of $T\partial M$ into $TM|_{\partial M}$.

Lemma 18.3.22. *If an orientation of M is represented by the section s of Λ^kTM, then there is a unique orientation of M which is represented by a section $\bar{s}$ of $\Lambda^{k-1}T\partial M$ satisfying $n \wedge \bar{s} = s$.*

Proof. For each $p \in \partial M$, fix a basis $e_1, \ldots, e_{k-1}$ of $T_p\partial M$. By adding n_p we get a basis of T_pM. Then $\bar{s}(p)$ is by definition $\bar{c}(p) \cdot e_1 \wedge \cdots \wedge e_{k-1}$ for some $\bar{c} \in \mathbb{R}$, and $s(p)$ similarly is $c(p) \cdot n_p \wedge e_1 \wedge \cdots \wedge e_{k-1}$ for some $c(p) \in \mathbb{R}$. From the equation

$$n_p \wedge (\bar{c}(p) \cdot e_1 \wedge \cdots \wedge e_{k-1}) = c(p) \cdot n_p \wedge e_1 \wedge \cdots \wedge e_{k-1}$$

we read off $\bar{c}(p) = c(p)$, so $\bar{s}$ is uniquely determined by n and s.

Firstly $\bar{s}$, up to multiplication by a positive smooth function, is independent of the choice of representative s: if s changes by multiplying it with positive smooth function, so does $\bar{s}$.

Next, we have to verify the orientation is independent of the choice of Riemannian metric g and smooth function χ. Modifying the latter just changes n by scalar multiplication by a positive smooth function, and hence has the same effect on $\bar{s}$. If we vary g, then n_p gets replaced by $n'_p = an_p + \sum_{i=1}^{k-1} a_i e_i$ with $a > 0$ so

$$n'_p \wedge (\bar{c}(p) \cdot e_1 \wedge \cdots \wedge e_{k-1}) = a \cdot n_p \wedge (\bar{c}(p) \cdot e_1 \wedge \cdots \wedge e_{k-1}),$$

and again $\bar{s}$ just changes by scalar multiplication by a positive smooth function. □

Definition 18.3.23. If M is oriented, we shall consider ∂M as oriented by the orientation produced in the previous lemma. We refer to this as the *induced orientation.*

Example 18.3.24. There is a preferred choice of orientation on $[0,1]$, namely using $1 \in \Lambda^1 T_p[0,1] \cong T_p[0,1] \cong \mathbb{R}$. Then

$$\partial[0,1] \cong \{1\} - \{0\},$$

where, for an oriented manifold N, $-N$ denotes the same manifold with opposite orientation.

More generally, if M is oriented without boundary, then

$$\partial([0,1] \times M) = M \times \{1\} - M \times \{0\}.$$

However, if we do $\partial(M \times [0,1])$ we get $(-1)^{\dim(M)}(M \times \{1\} - M \times \{0\})$. This is an unfortunate clash of our conventions for orientations and notation for homotopies.

Example 18.3.25. Generalizing Example 18.3.21 to the case that M has boundary and $f \pitchfork Z$, $\partial f \pitchfork Z$ we get that $\partial f^{-1}(Z) = (\partial f)^{-1}(Z)$ comes with two orientations: one as the boundary of an oriented manifold and one as the inverse image of an oriented manifold. These are not equal but satisfy

$$\partial f^{-1}(Z) = (-1)^{\operatorname{codim}(Z)} (\partial f)^{-1}(Z).$$

18.4 Integral intersection theory

Chapter 3 of [GP10] upgrades the mod 2 intersection theory to an integral version. The main input is the observation that

$$\partial[0,1] \cong \{1\} - \{0\}$$

and the classification of compact 1-dimensional manifolds lead to the following result:

Proposition 18.4.1. *If M is a compact oriented 1-dimensional manifold, then the number of positively-oriented points in ∂M is equal to the number of negatively-oriented points.*

So we can define intersection numbers with values in $\mathbb{Z}$ instead of $\mathbb{Z}/2$:

Definition 18.4.2. Suppose that Y is a compact oriented manifold without boundary, M is an oriented manifold and $Z \subset M$ is an oriented submanifold such that $\dim(Y) + \dim(Z) = \dim(M)$.

Let $f_0 \colon Y \to M$ be a smooth map. Then the *intersection number* $I(f_0, Z)$ is defined as follows: take f_1 homotopic to f_0 and transverse to Z, and set

$$I(f_0, Z) = \sum_{p \in f_1^{-1}(Z)} \textit{orientation of } p.$$

One proceeds as before, using Proposition 18.4.1 in place of the fact that the number of points in the boundary of a compact 1-dimensional manifold is even, to prove that $I(f_0, Z)$ is well-defined and establish its basic properties. You can then easily define integral versions of the degree of a map and the winding numbers, and use these to great effect.

Example 18.4.3. With these definitions in hand, the mod 2 linking numbers of Section **??** generalize to integer linking numbers.

18.5 Problems

Problem 46 (Codimension 1 submanifolds are orientable)**.** Use the Jordan–Brouwer separation theorem to prove that if $M \subset \mathbb{R}^k$ is a compact codimension 1 submanifold, then it is orientable.

Problem 47. Use partitions of unity to prove that any vector $v \in T_pM$ is the value at x of some smooth vector field X on M.

Chapter 19

Differential forms and integration

Today we define differential forms and one of their raisons-d'etre: integration. This is Section 4.§4 of [GP10], but you should also take a look at Sections 4.§1–3 if you haven't done so already.

19.1 Differential forms

We start with a discussion of differential forms, with a focus of forms of top degree.

19.1.1 The definition of differential forms

The definition of 1-forms

Every smooth manifold has a tangent bundle TM, which you are already familiar with, and a *cotangent bundle* T^*M. The fibers T_p^*M of the cotangent bundle, called cotangent spaces, are the linear duals $(T_pM)^*$ of the tangent spaces. If M has dimension k, both are smooth vector bundles of dimension k.

Definition 19.1.1. A *1-form* on M is a smooth section of T^*M.

We can produce a 1-form from a smooth function $f\colon M \to \mathbb{R}$. Recall that the fibres T_mM of the tangent bundle are derivations of germs $\mathcal{E}(M,m)$ near m of smooth functions $M \to \mathbb{R}$. In particular, these assign a number to each the germ $\overline{f}$ of f. We get an element $(df)_m$ of $(T_mM)^*$ by taking

$$\begin{aligned} df\colon T_mM &\longrightarrow \mathbb{R} \\ X &\longmapsto X(\overline{f}). \end{aligned}$$

This produces an element of T_mM for each m, hence a section. To see it is smooth we use charts:

Example 19.1.2. If $\phi\colon \mathbb{R}^k \supset U \to V \subset M$ is a chart around $p \in M$, its derivative induces an isomorphism of $T_{\phi^{-1}(p)}\mathbb{R}^k$ with T_pM. The former, one thinks of as the $\mathbb{R}$-vector space with basis given by partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ (this is just alternative notation for the standard basis vector $e_1, \ldots, e_k$, but now considered as elements of $T_{\phi^{-1}(p)}\mathbb{R}^k \cong \mathbb{R}^k$). This in turn gives rise to a dual basis $dx_1, \ldots, dx_k$ of T_p^*M. Thus every 1-form α can be written in local coordinates as

$$\alpha(x) = \sum_{i=1}^{k} a_i(x) dx_i.$$

We saw above that any smooth function $f\colon M \to \mathbb{R}$ gives rise to a 1-form df. In terms of the above coordinates this is given by

$$T_p^*M \ni (df)_p := \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(p) dx_i.$$

To see this, observe that $(df)_p$ by construction evaluates on $\frac{\partial}{\partial x_i}$ to $\frac{\partial f}{\partial x_i}(p)$.

Example 19.1.3. The 1-form $-ydx + xdy$ on $\mathbb{R}^2$ restricts to a 1-form on $S^1 \subset \mathbb{R}^2$ which is nowhere-vanishing.

The definition of p-forms

We extend the notion of a 1-form to a p-form as follows:

Definition 19.1.4. A *p-form* is a smooth section of $\Lambda^p T^*M$.

Example 19.1.5. As $\Lambda^0 T^*M = \mathbb{R}$, a smooth 0-form is a smooth function. As $\Lambda^1 T^*M = T^*M$, this recovers the definition of a smooth 1-form given above.

Since the value at $p \in M$ of a smooth section of a smooth vector bundle E lie in $\mathbb{R}$-vector spaces E_p, so we can define addition of smooth sections by pointwise addition. Similarly, we can scale a smooth section with any smooth real-valued function. The result is either operation is again smooth section, making the set $\Gamma(M, E)$ into a $C^\infty(M;\mathbb{R})$-module. Since $C^\infty(M;\mathbb{R})$ contains $\mathbb{R}$ as the constant functions, $\Gamma(M, E)$ is in particular an $\mathbb{R}$-vector space.

Definition 19.1.6. $\Omega^p(M)$ is the $\mathbb{R}$-vector space $\Gamma(M, \Lambda^p T^*M)$ of p-forms.

Definition 19.1.7. $\Omega^*(M)$ is the graded $\mathbb{R}$-vector space of differential forms on M, given by putting the p-forms $\Omega^p(M)$ in degree p. When the degree plays no role, we refer to a p-form as a *differential form of degree p*.

Recalling that M is k-dimensional, we see that $\Lambda^p T^*M = 0$ if $p > k$, and hence there are no non-zero differential forms of degree larger than the dimension of M. In this lecture our main interest is the case $p = k$. Then $\Lambda^k T^*M$ is one-dimensional, and we shall refer to the k-forms as *top forms*.

Example 19.1.8. A chart $\phi\colon \mathbb{R}^k \supset U \to V \subset M$ induces a local trivialization of TM. In turn, this gives a local trivializations of T^*M and hence of $\Lambda^p T^*M$. For this we see that each p-form $\omega \in \Omega^p(V)$ can be written in local coordinates as

$$\omega(x) = \sum_I a_I(x) dx_I$$

where for each index set $I = 1 \leqslant i_1 < \ldots < i_p \leqslant k$, we write

$$dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

and $a_I \colon U \to \mathbb{R}$ is a some smooth function. In particular, every top form can be written in local coordinates as

$$\omega(x) = a(x) dx_1 \wedge \cdots \wedge dx_k$$

for a smooth function $a \colon U \to \mathbb{R}$.

Example 19.1.9. Recall that an orientation was an everywhere non-vanishing section of $\Lambda^k TM$, up to scaling by everywhere positive function. Recall that a Riemannian metric is a smooth family of non-degenerate bilinear forms on TM, and always exists. Such a Riemannian metric gives an isomorphism of TM and T^*M by sending a vector $v \in T_pM$ to the linear functional $w \mapsto \langle w, v \rangle$ in T_p^*M. This isomorphism induces an isomorphism between the line bundles $\Lambda^k TM$ and $\Lambda^k T^*M$, and hence an orientation is also the same as an everywhere non-vanishing top form up to scaling.

19.1.2 The wedge product

We defined a wedge product

$$\wedge \colon \Omega^p(M) \otimes \Omega^q(M) \to \Omega^{p+q}(M),$$

induced by the corresponding wedge product on the exterior powers of the fiber. This has the following property:

Theorem 19.1.10. *The wedge product makes $\Omega^*(M)$ into a graded-commutative $\mathbb{R}$-algebra. That is, the wedge product has the following properties:*

(1) It is unital with unit given by the function that is constant 1.

(2) It is bilinear.

(3) It is associative.

(4) If ω has degree p and ρ has degree q, then $\omega \wedge \rho$ has degree $p+q$ and

$$\omega \wedge \rho = (-1)^{pq} \rho \wedge \omega$$

Remark 19.1.11. Observe that $\Omega^0(M) = C^\infty(M;\mathbb{R})$, and the wedge product $\Omega^0(M) \otimes \Omega^p(M) \to \Omega^p(M)$ is equal to the multiplication of the $C^\infty(M;\mathbb{R})$-module structure. Hence we can replace linearity by $C^\infty(M;\mathbb{R})$-linearity.

We can use the wedge products to produce many top forms, e.g. by wedging together k 1-forms as below:

Example 19.1.12. Given k smooth functions $f_1, \ldots, f_k \colon \mathbb{R}^k \to \mathbb{R}$, we can produce a top form

$$df_1 \wedge \cdots \wedge df_k,$$

whose value in local coordinates is given by

$$(df_1 \wedge \cdots \wedge df_k)_p = \det\left(\frac{\partial f_j}{\partial x_i}\right) dx_1 \wedge \cdots \wedge dx_k.$$

If you don't see how to do this computation, please ask about it in office hours or sections.

19.1.3 Pullback of differential forms

One of the advantages of differential forms is that we can pull them back along any smooth map, unlike vector fields, which can only be pushed forward along a diffeomorphism:

Theorem 19.1.13. *Each smooth map $f\colon M \to N$ induces a map $f^*\colon \Omega^*(N) \to \Omega^*(M)$ of graded-commutative $\mathbb{R}$-algebras by applying to p-forms the map $\Lambda^p(d_pf)^*$ in each fiber. Pullback has the following properties:*

(1) On functions (that is, 0-forms), f^* is given by precomposition, $f^*g = g \circ f$.

(2) $(g \circ f)^* = f^* \circ g^*$ and $(\mathrm{id})^* = \mathrm{id}$.

(3) It commutes with wedge products:

$$f^*(\omega \wedge \rho) = f^*(\omega) \wedge f^*(\rho).$$

(4) It commutes with taking derivatives of functions:

$$f^*dg = d(f^*g).$$

Example 19.1.14. Let's compute some pullback in local coordinates. Suppose $f\colon \mathbb{R}^k \supset U \to V \subset \mathbb{R}^{k'}$ is a smooth map and recall that $dx'_i \in \Omega^1(V)$ is the dual to the vector field $\frac{\partial}{\partial x'_i}$ that is constant equal to e'_i. Then

$$f^*dx'_i = df^*x'_{i'} = df_{i'} \sum_{i=1}^{k} \frac{\partial f_{i'}}{\partial x_i} dx_i,$$

with $f_{i'}$ the i'th component of f.

A similar formula exists for p-forms, but we will focus on the case of top forms. Suppose a p-form $\omega \in \Omega^p(V)$ is given by

$$\omega(x') = a(x')dx'_1 \wedge \cdots \wedge dx'_{k'}.$$

Since pullback commutes with wedge product, its pullback $f^*\omega \in \Omega^p(V')$ must given by

$$\begin{aligned} f^*\omega(x) &= a(f(x))f^*(dx'_1 \wedge \cdots \wedge dx'_{k'}) \\ &= a(f(x))f^*(dx'_1) \wedge \cdots \wedge f^*(dx'_{k'}) \end{aligned}$$

and above we saw how to compute each term $f^*(dx'_{i'})$ in terms of the partial derivatives of $f_{i'}$.

Given a submanifold $X \subset M$ with inclusion denoted $i\colon X \hookrightarrow M$, we can restrict a p-form $\omega \in \Omega^p(M)$ to X:

$$\Omega^*(M) \ni \omega \longmapsto i^*\omega \in \Omega^p(X).$$

If X is p-dimensional, this gives a top form on X.

Example 19.1.15. The restriction to $S^1 \subset \mathbb{R}^2$ of the 1-form $\omega = xdx + ydy$ is identically 0. This is because that ω is dg with $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\frac{1}{2}(x^2 + y^2)$. Hence $i^*\omega = i^*dg = d(i^*g)$ and since i^*g constant equal to 1 its derivative vanishes.

19.2 Integration of differential forms

Our next goal is the integration of k-forms over k-dimensional manifolds. This theory of integration has a number of features which may differ from what you are used to:

(I) It is only defined for oriented manifolds.

(II) Over a k-dimensional oriented manifold, you can only integrate top forms (so only k-forms, not functions).

(III) We will only define integration of compactly-supported top forms.

19.2.1 Integration on $\mathbb{R}^k$

First suppose that

$$\omega = a(x)\, dx_1 \wedge \cdots \wedge dx_k,$$

is a top form on an open subset $U \subset \mathbb{R}^k$. Then, as the notation suggests, we shall define

$$\int_U \omega = \int_U a(x)\, dx_1 \wedge \cdots \wedge dx_k := \int_U a(x)\, dx_1 \cdots dx_k,$$

and to guarantee that the integral exist we assume a has compact support in U. This is not really necessary as some integrals of functions without compact support do converge, but it is the only case we shall use. For smooth compactly-supported functions both the Riemann and Lebesgue integral exist and are equal, so we don't need to worry about the technical details too much.

Example 19.2.1. The order of the entries of $dx_1 \wedge \cdots \wedge dx_k$ is important: if $U = \mathrm{int}(D^2)$ and $\omega = dy \wedge dx$, then (ignoring the compact support requirement)

$$\int_U \omega = \int_{\mathrm{int}(D^2)} dy \wedge dx = -\int_{\mathrm{int}(D^2)} dx \wedge dy = -\pi.$$

How does the integral of a top form transform under a change of coordinates? That is, suppose we have a diffeomorphism $\psi\colon U' \to U$. Then on the one hand, we have by definition of the integral that

$$\begin{aligned}
\int_{U'} \psi^*\omega &= \int_{U'} \psi^* a(x')\, \psi^* dx_1 \wedge \cdots \wedge \psi^* dx_k \\
&= \int_{U'} (a \circ \psi)(x')\, d\psi_1 \wedge \cdots \wedge d\psi_k \\
&= \int_{U'} (a \circ \psi)(x')\, \det\left(\frac{\partial \psi_j}{\partial x'_i}\right) dx'_1 \wedge \cdots \wedge dx'_k,
\end{aligned}$$

and recognizing the matrix that we are taking the determinant of as the total derivative of ψ, we get

$$\int_{U'} \psi^*\omega = \int_{U'} (a \circ \psi)(x')\, \det(D_{x'}\psi)\, dx'_1 \cdots dx'_k. \tag{19.1}$$

On the other hand, the change-of-variables formula from multivariable calculus [DK04b, Theorem 6.6.1] says:

Theorem 19.2.2. *With notation as above,*

$$\int_U a(x)dx_1 \cdots dx_k = \int_{U'} (a \circ \psi)(x') \left|\det(D_{x'}\psi)\right| dx'_1 \cdots dx'_k. \tag{19.2}$$

Remark 19.2.3. To see that the absolute values signs belong in this formula, observe that in the integral you use the values of a and the volumes of blocks, without a sign.

That is, (19.1) and (19.2) could differ by a sign (or even worse if U has many components) and to avoid this, we have to understand when the sign of $\det(D_{x'}\psi)$ is positive. This determinant also appears as the multiple of $e_1 \wedge \cdots \wedge e_k$ one obtains when applying

$$\Lambda^k(D_{x'}\psi)\colon \Lambda^k T_{x'}U' \cong \mathbb{R}\cdot(e_1 \wedge \cdots \wedge e_k) \longrightarrow \Lambda^k T_x U \cong \mathbb{R}\cdot(e_1 \wedge \cdots \wedge e_k)$$

to $e_1 \wedge \cdots \wedge e_k$. We said that ψ *preserves orientation* if this multiple is positive. That is, we conclude the following:

Corollary 19.2.4. *If $\omega \in \Omega^k(U)$ is a compactly-supported top form and $\psi\colon \mathbb{R}^k \supset U' \to U \subset \mathbb{R}^k$ is an orientation-preserving diffeomorphism, then*

$$\int_{U'} \psi^*\omega = \int_U \omega.$$

19.2.2 Integration on manifolds

We shall define the integral of a compactly-supported top form ω over an oriented manifold M in several steps.

Theorem 19.2.5. *There is a unique construction of an integral of top forms on oriented k-dimensional manifolds with the following properties:*

(1) If the manifold has an orientation-preserving diffeomorphism to an open subset of $\mathbb{R}^k$, it is the integral defined above (note that this is independent of the choice of such diffeomorphism by Corollary 19.2.4).

(2) If ω is supported in $U \subset M$ then $\int_M \omega = \int_U \omega$.

(3) It is linear.

Proof. Desiderata (1) and (2) imply that if ω happens to be supported in the image of an orientation-preserving chart $\phi\colon \mathbb{R}^k \supset U \to V \subset M$ (using the standard orientation on U_α inherited from $\mathbb{R}^k$), we must define

$$\int_V \omega := \int_U \phi^*\omega.$$

If M is oriented, we can find an open cover of M by charts $\phi_\alpha\colon \mathbb{R}^k \supset U_\alpha \to V_\alpha \subset M$ so that all transition functions are orientation-preserving. Now pick a partition of unity η_α subordinate to the V_α, and observe that $\omega = \sum_\alpha \eta_\alpha\omega_\alpha$

which is a finite sum because the support of ω is compact. Thus desideratum (3) forces us to define

$$\int_M \omega := \sum_\alpha \int_{V_\alpha} \eta_\alpha \omega_\alpha,$$

which makes sense because it is a finite sum.

We need to verify that this is independent of the choice of open cover and partition of unity. Take a second collection of charts $\phi'_\beta \colon \mathbb{R}^k \supset U'_\beta \to V'_\beta \subset M$ and a subordinate partition of unity ρ'_β. Using the fact that $\sum_\beta \rho'_\beta = 1$ and the sums are finite so may be interchanged, we get

$$\begin{aligned}\sum_\alpha \int_{V_\alpha} \eta_\alpha \omega &= \sum_\alpha \int_{V_\alpha} \eta_\alpha \Big(\sum_\beta \rho'_\beta \omega\Big) \\ &= \sum_\alpha \sum_\beta \int_{V_\alpha \cap V'_\beta} \eta_\alpha \rho'_\beta \omega,\end{aligned}$$

and by symmetry this is also $\sum_\beta \int_{V'_\beta} \rho'_\beta \omega$. □

Example 19.2.6. If $-M$ denotes M with opposite orientation, then $\int_{-M} \omega = -\int_M \omega$.

Example 19.2.7. If ω is a p-form for $p < k$, we can't integrate it over the k-dimensional manifold M. However, we can integrate it over an oriented submanifold $X \subset M$ of dimension p:

$$\int_X \omega := \int_X i^* \omega$$

with $i \colon X \hookrightarrow M$ the inclusion.

Remark 19.2.8. From the construction in Theorem 19.2.5, we see that if you have a preferred collection of $\{(U_i, V_i, \phi)\}$ of M such that $\bigcup_i V_i = M$, you can use only these charts in your construction of the integral.

Using this definition, Corollary 19.2.4 generalizes to manifolds:

Corollary 19.2.9. *If $f \colon M \to N$ is an orientation-preserving diffeomorphism and $\omega \in \Omega^k(N)$ is a compactly-supported top form, then*

$$\int_M f^* \omega = \int_N \omega.$$

This definition of the integral is useful for proving theorems, but hard to use in practical computations. In practice one does the following. We start with two observations: the above construction goes through for Riemann-integrable forms, not just smooth ones, and for manifolds with boundary (or even corners).

Now suppose one has a finite collection of orientation-preserving embeddings $\varphi_i \colon \mathbb{R}^k \supset N_i \to M$ of submanifolds with boundary (or even corners), which only intersect at their boundary. Then we can decompose a smooth ω as a finite sum of Riemann-integrable forms $\sum_i 1_{\varphi_i(N_i)} \omega$ with $1_{\varphi_i(N_i)}$ the indicator function of $\varphi_i(N_i)$, and evaluate the integral as

$$\int_M \omega = \sum_i \int_{\varphi_i(N_i)} 1_{\varphi_i(N_i)} \omega = \sum_i \int_{N_i} \varphi_i^* \omega.$$

Example 19.2.10. This tells you that to compute the integral of a 2-form over S^2, you decompose S^2 into the two hemisphere, parametrize these by a disk, and you take the sum of the values of the integral of the pullback of the 2-form to both disks. In other words, it's what you have been doing in multivariable calculus all along.

Chapter 20

The exterior derivative and Stokes' theorem

Stokes theorem is a generalization of the formula

$$\int_0^1 \frac{\partial f}{\partial x} dx = f(1) - f(0).$$

To state it, we first need to generalize the derivative of a function to differential forms; the *exterior derivative.* The proof of Stokes' theorem will then follow from an easily proven version in charts. This material can be found in Sections 4.§5 and 4.§7 of [GP10].

20.1 The exterior derivative

As for the integral, we shall first define the exterior derivative on open subsets of $\mathbb{R}^k$ and then extend it to arbitrary smooth manifolds using charts. Suppose we are given a p-form on an open subset $U \subset \mathbb{R}^k$,

$$\omega = \sum_I a_I dx_I,$$

the sum ranging over all $1 \leqslant i_1 < \ldots < i_p \leqslant k$ and $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then the exterior derivative is given by taking the ith partial derivative of each of the coefficients and wedging with dx_i:

$$d\omega = \sum_i \sum_I \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I.$$

Some of the terms in this sum vanish, when i is among the indexing set I. More generally, signs appears when shuffling dx_i into its standard position.

Example 20.1.1. If $f \colon \mathbb{R}^3 \supset U \to \mathbb{R}$ is a smooth function, i.e. a 0-form, then its exterior derivative is

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3.$$

This coincides with the definition we used before. This is related to the *gradient* of the function f.

Example 20.1.2. If we have a 1-form on $U \subset \mathbb{R}^3$,

$$\alpha = a_1 dx_1 + a_2 dx_2 + a_3 dx_3,$$

then its exterior derivative is

$$\begin{aligned} d\alpha = & \left(\frac{\partial a_1}{\partial x_1} dx_1 + \frac{\partial a_1}{\partial x_2} dx_2 + \frac{\partial a_1}{\partial x_3} dx_3\right) \wedge dx_1 \\ & + \left(\frac{\partial a_2}{\partial x_1} dx_1 + \frac{\partial a_2}{\partial x_2} dx_2 + \frac{\partial a_2}{\partial x_3} dx_3\right) \wedge dx_2 \\ & + \left(\frac{\partial a_3}{\partial x_1} dx_1 + \frac{\partial a_3}{\partial x_2} dx_2 + \frac{\partial a_3}{\partial x_3} dx_3\right) \wedge dx_3 \\ = & \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right) dx_1 \wedge dx_3 + \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) dx_2 \wedge dx_3. \end{aligned}$$

This is related to the *curl* of the vector field with components (a_1, a_2, a_3).

Example 20.1.3. If we have a 2-form on $U \subset \mathbb{R}^3$,

$$\omega = a_1 dx_2 \wedge dx_3 - a_2 dx_1 \wedge dx_3 + a_3 dx_1 \wedge dx_2,$$

then its exterior derivative is

$$d\omega = \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3.$$

This is related to the *divergence* of the vector field with components (a_1, a_2, a_3).

The exterior derivative has the following properties, and the following also serves as a definition:

Theorem 20.1.4. *The exterior derivative is the unique operation $\Omega^*(U) \to \Omega^{*+1}(U)$ with the following properties:*

(i) For smooth functions $f \in \Omega^0(U)$, $df = \sum_{i=1}^k \frac{\partial f}{\partial x_i} dx_i$.

(ii) It is linear, $d(\omega + \nu) = d\omega + d\nu$.

(iii) It is a graded derivation for the wedge product, for a p-form ω and a q-form ν, $d(\omega \wedge \nu) = d(\omega) \wedge \nu + (-1)^p \omega \wedge d(\nu)$.

(iv) It is a differential, $d(d\omega) = 0$.

Proof. We first verify that the exterior derivative satisfies the above properties. Property (i) is true by definition, and property (ii) follows from the fact that partial derivatives are linear. Properties (iii) and (iv) are slightly harder; the former is essentially the product rule and the latter the fact that partial derivatives commute.

By linearity of d and the fact that $\wedge$ distributes over finite sums, it suffices to prove (iii) in the case that $\omega = a_I dx_I$ and $\nu = b_J dx_J$. Then $\omega \wedge \nu = a_I b_J dx_I \wedge dx_J$

and we have that

$$\begin{aligned}
d(\omega \wedge \nu) &= \sum_{i=1}^{k} \frac{\partial(a_I b_J)}{\partial dx_i} dx_i \wedge dx_I \wedge dx_J \\
&= \sum_{i=1}^{k} \left(\frac{\partial a_I}{\partial dx_i} b_J dx_i \wedge dx_I \wedge dx_J + a_I \frac{\partial b_J}{\partial dx_i} dx_i \wedge dx_I \wedge dx_J \right) \\
&= \left(\sum_{i=1}^{k} \frac{\partial a_I}{\partial dx_i} dx_i \wedge dx_I \right) \wedge (b_J dx_J) + (-1)^{pq} a_I dx_I \wedge \left(\sum_{i=1}^{k} \frac{\partial b_J}{\partial dx_i} dx_i \wedge dx_J \right) \\
&= d(\omega) \wedge \nu + (-1)^p \omega \wedge d(\nu).
\end{aligned}$$

Similarly, it suffices to prove (iv) in the case that $\omega = a_I dx_I$. Since $d(dx_i) = 0$ (we are only taking partial derivatives of constant functions), we can use (iii) twice to write

$$d(d(a_I dx_I)) = d(da_I) \wedge dx_I,$$

and hence it suffices to show that $d(da_I) = 0$. But we have

$$d(da_I) = \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_i \wedge dx_j = \sum_{1 \leqslant i < j \leqslant k} \frac{\partial^2 a_I}{\partial x_i \partial x_j} (dx_i \wedge dx_j + dx_j \wedge dx_i) = 0.$$

Here we have first used that partial derivatives of smooth functions commute, and that $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j + dx_j \wedge dx_i = dx_i \wedge dx_j - dx_i \wedge dx_j = 0$.

The next goal is to prove uniqueness. Suppose that $D \colon \Omega^*(U) \to \Omega^{*+1}(U)$ satisfies the same property, then we must show that $d = D$. But if we try to prove d and D coincide on a general p-form

$$\omega = \sum_I a_I dx_I,$$

then by (ii) it suffices to prove they coincide on $a_I dx_I$. By (iii) it then suffices to prove they coincide on a_I and each dx_i. By (i), they indeed coincide on a_I. For dx_i, observe that $dx_i = d(x_i)$ which equals $D(x_i)$ by (i), so that by (iv) $d(dx_i) = 0$ and $D(dx_i) = D(D(x_i)) = 0$. □

The exterior derivative commutes with pullback:

Proposition 20.1.5. *If $g \colon U' \to U$ is a smooth map between open subsets of Euclidean spaces, then $g^* d = dg^*$.*

When g is a diffeomorphism, there is an elegant proof by observing that $(g^{-1})^* dg^*$ has the same properties as d, so by uniqueness of the exterior derivative has to be equal to it.

Proof. Recall that g^* has the following properties: (i') $g^* df = d(f \circ g)$, (ii') it is linear, (iii') it commutes with wedge product. These formal properties imply the proposition as follows: to prove that $g^* d$ and dg^* coincide on a general p-form

$$\omega = \sum_I a_I dx_I,$$

by (ii) and (ii') it suffices to prove they coincide on each $a_I dx_I$. Then by (iii) and (iii') it suffices to prove they coincide on a_I and each dx_I. Property (i') says they coincide on a_I. For dx_i, we observe that $g^* d(dx_i) = g^* 0 = 0$ and to prove that the other side also vanishes we write $g^* dx_i = g^* d(x_i) = dg$ so $dg^* dx_i = d^2 g = 0$. $\square$

Since d in particular commutes with pullback along a diffeomorphism, we can extend to smooth manifolds of dimension k using charts. For $\omega \in \Omega^p(M)$, $d\omega$ is defined near a point in M by picking a chart $\phi\colon \mathbb{R}^k \supset U \to V \subset M$ and taking $(\phi^{-1})^* d\phi^*\omega$. The previously established properties all generalize to manifolds, as they can be verified in a chart. This theorem serves as the definition of the *exterior derivative*.

Theorem 20.1.6. *There is an operation $d\colon \Omega^*(-) \to \Omega^{*+1}(-)$ on differential forms on manifolds uniquely determined by the following properties:*

- *(i) On smooth functions, i.e. 0-forms, it is the ordinary derivative.*
- *(ii) It is linear.*
- *(iii) It is a derivation for the wedge product.*
- *(iv) It is a differential, $d^2 = 0$.*
- *(v) It commutes with pullbacks along smooth maps.*

We also add one useful observation from the point of view of integration: if ω is compactly-supported so is $d\omega$. Letting $\Omega^*_c(-)$ denote the *compactly-supported forms*, we can restrict d to an operation $\Omega^*_c(-) \to \Omega^{*+1}_c(-)$. Note that the pullback of a compactly-supported form is *not* in general compactly-supported; this requires the map to the proper as $\mathrm{supp}(g^*\omega) \subset g^{-1}(\mathrm{supp}(\omega))$ and properness is exactly the condition that the inverse image of a compact subset is compact.

20.2 Stokes' theorem

Recall that last lecture we defined the integral of a compactly-supported top form over an oriented manifold, using partitions of unity.

Theorem 20.2.1 (Stokes)**.** *Let $\omega \in \Omega^{k-1}_c(M)$ be a compactly-supported $(k-1)$-form on an oriented smooth manifold M of dimension k with boundary ∂M, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

In this theorem, we need ∂M to be oriented as well and to get this equation to hold, we use our convention for the induced orientation on the boundary ("outward-pointing first").

Example 20.2.2. Let $M = [0,1]$ with its standard orientation. Then $\partial[0,1] = \{0,1\}$, where 1 has positive orientation and 0 has negative orientation. In this case Stokes concerns 0-forms, i.e. functions, and says

$$\int_0^1 \frac{\partial f}{\partial x} dx = \int_{[0,1]} df = \int_{\partial[0,1]} f = f(1) - f(0),$$

a formula that should be quite familiar.

In fact, our proof will use the above result as input, a basic result in single-variable calculus. We also use Fubini's theorem on successive integration, e.g. [DK04b, Theorem 6.4.5].

Proof. Pick an open cover of M by the codomains V_α of a collection of charts $\phi_\alpha\colon [0,\infty)\times\mathbb{R}^{k-1}\supset U_\alpha\to V_\alpha\subset M$. Also pick a subordinate partition of unity $\eta_\alpha\colon M\to[0,1]$. Then $\omega=\sum_\alpha \eta_\alpha\omega$, and this sum is finite because $\operatorname{supp}(\omega)$ is compact. Since both $\int_M$ and d are linear, we may thus assume that ω is supported in V_α. Then so is $d\omega$ and we have

$$\int_M d\omega=\int_{U_\alpha}\phi_\alpha^* d\omega \qquad\text{and}\qquad \int_{\partial M}\omega=\int_{\partial U_\alpha}\phi_\alpha^*\omega.$$

Since ϕ_α^* commutes with d, we might as well replace $\phi_\alpha^*\omega$ by ω to simplify notation and extend this by 0 to a compactly-supported $(k-1)$-form on $[0,\infty)\times\mathbb{R}^{k-1}$. We have thus reduced our task to proving Stokes theorem in the special case $M=[0,\infty)\times\mathbb{R}^{k-1}$.

Since both $\int_{[0,\infty)\times\mathbb{R}^{k-1}} d\omega$ and $\int_{\{0\}\times\mathbb{R}^{k-1}}\omega$ are linear in ω, it suffices to prove this for $\omega=a dx_I$ with $I=1<\ldots<\hat{i}<\ldots<k$. Then $d\omega=(-1)^{i-1}\frac{\partial a}{\partial x_i}dx_1\wedge\cdots\wedge dx_k$. There are two cases: (i) $i=1$, (ii) $i>1$.

Let's start with the latter. Then ω restricts to 0 on ∂M (as it contains a dx_1) so we should get 0. Pick N sufficiently large so that $\operatorname{supp}(a_I)\subset[0,N]\times[-N,N]^{k-1}$, then

$$\begin{aligned}
\int_{[0,\infty)\times\mathbb{R}^{k-1}} d\omega &= \int_{[0,N]\times[-N,N]^{k-1}}(-1)^{i-1}\frac{\partial a}{\partial x_i}dx_1\wedge\cdots\wedge dx_k\\
&=\int_{[0,N]\times[-N,N]^{k-2}}\left(\int_{[-N,N]}\frac{\partial a}{\partial x_i}dx_i\right)dx_I\\
&=\int_{[0,N]\times[-N,N]^{k-2}}\left(a(x_1,\ldots,N,\ldots,x_k)-a(x_1,\ldots,-N,\ldots,x_k)\right)dx_I\\
&=0=\int_{\{0\}\times\mathbb{R}^{k-1}}\omega.
\end{aligned}$$

Here we have used Fubini's theorem, the fundamental theorem of analysis and that a is supported in $[0,N]\times[-N,N]^{k-1}$ so that both $a(x_1,\ldots,N,\ldots,x_k)$ and $a(x_1,\ldots,-N,\ldots,x_k)$ are 0.

The former case is similar, but has a different outcome. Pick N as before, then

$$\begin{aligned}
\int_{[0,\infty)\times\mathbb{R}^{k-1}} d\omega &= \int_{[0,N]\times[-N,N]^{k-1}} \frac{\partial a}{\partial x_1} dx_1 \wedge \cdots \wedge dx_k \\
&= \int_{[-N,N]^{k-1}} \left(\int_{[0,N]} \frac{\partial a}{\partial x_1} dx_1 \right) dx_I \\
&= \int_{[-N,N]^{k-1}} \left(a(N, x_2, \ldots, x_k) - a(0, x_2 \ldots, x_k)\right) dx_I \\
&= -\int_{[-N,N]^{k-1}} a(0, x_2, \ldots, x_k) dx_2 \wedge \cdots \wedge dx_k \\
&= \int_{\{0\}\times\mathbb{R}^{k-1}} \omega.
\end{aligned}$$

Here we have used the same tools as before, as well as $a(N, x_2, \ldots, x_k) = 0$. Our convention on the orientation of the boundary was chosen exactly so that the signs cancel in the last step: in the "outward-pointing first convention", a basis $(v_2, \ldots, v_k)$ of $T_x(\{0\} \times \mathbb{R}^{k-1})$ is positively oriented if $(-e_1, v_2, \ldots, v_k)$ is, that is $(-e_1) \wedge v_2 \wedge \cdots \wedge v_k$ equals $e_1 \wedge \cdots \wedge e_k$ up to scaling by a positive real number. Hence the induced orientation on $\{0\} \times \mathbb{R}^{k-1}$ as a boundary of the upper half-plane is opposite to the usual orientation. □

We now give a number of applications.

20.2.1 Integrating pullbacks

Suppose that W is a oriented smooth manifold of dimension k with boundary ∂W and $f \colon W \to M$ is a smooth map. Then if a compactly-supported $(k-1)$-form ω satisfies $d\omega = 0$, we get that

$$\int_W df^*\omega = \int_W f^*(d\omega) = 0,$$

but applying Stokes' formula we also get

$$\int_W df^*\omega = \int_{\partial W} f^*\omega.$$

In particular, if ∂W comes divided into a disjoint union $\partial_{\mathrm{in}} W \sqcup \partial_{\mathrm{out}} W$ we may artificially reverse the orientation on $\partial_{\mathrm{in}} W$ (so it is "inward pointing first") and get the formula

$$\int_{\partial_{\mathrm{out}} W} f^*\omega - \int_{\partial_{\mathrm{in}} W} f^*\omega = 0.$$

We will use the following consequence in the next lecture:

Corollary 20.2.3. *If f_0 and f_1 are homotopic smooth maps $X \to M$ with X of compact dimension k and $\omega \in \Omega^k(M)$ satisfies $d\omega = 0$, then*

$$\int_X f_1^*\omega = \int_X f_0^*\omega$$

for all compactly-supported k-forms ω.

Proof. Suppose $W = X \times [0,1]$, $\partial_{\text{in}} W = X \times \{0\}$, $\partial_{\text{out}} W = X \times \{1\}$. Then we can think of $f\colon X \times [0,1] \to M$ as a homotopy from $f_0 := f|_{X \times \{0\}}$ to $f_1 := f|_{X \times \{1\}}$. The orientation on $X \times \{0\}$ and $X \times \{1\}$ are now equal (instead of opposite, if we had taken the usual convention) and we get the equation

$$\int_X f_1^* \omega = \int_{\partial_{\text{out}} W} f^* \omega = \int_{\partial_{\text{in}} W} f^* \omega = \int_X f_0^* \omega.$$

Thus the integral of the pullback along f of a closed form only depend on the homotopy class of f. □

20.3 Classical integral theorems

We now explain how the multivariable calculus theorems you have learned are special cases of Stokes' theorem. This is significantly harder than one might expect, because the classical version is harder to state precisely. In particular, we have to make precise the notions of "line element," "surface element," and "volume element."

That is, we need to explain how to integrate continuous functions $f\colon X \to \mathbb{R}$ over a smooth submanifold X of Euclidean space. We will do following [DK04b, Chapter 7]. As for integrals of differential forms, we can not just integrate in charts due to the Jacobian term in the change-of-variables formula. To correct for this, we need a *density*:

Definition 20.3.1. A *density* for a manifold M is an assignment to each chart $(U_\alpha, V_\alpha, \phi_\alpha)$ of M a continuous function $\rho_\alpha\colon U_\alpha \to \mathbb{R}$ such that

$$\rho_\alpha(x) = \rho_\beta(\phi_{\alpha\beta}(x))|\det D_x \psi_{\alpha\beta}|.$$

We then define an integral of a continuous function $f\colon M \to \mathbb{R}$ analogously to the integral of differential forms. We pick a partition of unity $\eta_\alpha\colon M \to \mathbb{R}$ with respect of the codomains V_α of charts, and set

$$\int_M f d\rho := \sum_\alpha \int_{U_\alpha} \eta_\alpha(\phi_\alpha(x)) f(\phi_\alpha(x)) \rho_\alpha(x) dx_1 \cdots dx_k.$$

Definition 20.3.1 gives, by the same argument as in proof of that theorem, that this is well-defined (i.e. independent of η_α).

If $X \subset \mathbb{R}^k$ is a r-dimensional smooth submanifold, then there is a canonical choice of density, the *Euclidean density*: in this case we can make sense of the total derivative of $D_x\phi_\alpha$ as a $(k \times r)$-matrix, and set

$$\rho_\alpha^{\text{eucl}}(x) := \sqrt{\det((D_x\phi_\alpha)^t (D_x \phi_\alpha))}.$$

See [DK04b, Theorem 7.3.1] for a proof that this is a density.

The integrals of functions using "line elements," "surface elements," or "volume elements" are exactly those with respect to the Euclidean density. We will now identify integrals of differential forms as integrals of certain functions with respect to the Euclidean density.

Lemma 20.3.2. *Suppose that M is a compact codimension 0 submanifold of $\mathbb{R}^k$ with boundary ∂M, and $\omega \in \Omega^k(M)$. Define a smooth function $f\colon M \to \mathbb{R}$ by $\nu(x) = f(x)dx_1 \wedge \cdots \wedge dx_k$. We have*

$$\int_M \omega = \int_M f d\rho^{\text{eucl}}.$$

Lemma 20.3.3. *Suppose that M is a compact codimension 0 submanifold of $\mathbb{R}^k$ with boundary ∂M, and $\omega \in \Omega^{k-1}(M)$. Define a smooth vector field*

$$\vec{V}(x) = \begin{bmatrix} a_1(x) \\ \vdots \\ a_k(x) \end{bmatrix}$$

$\omega(x) = \sum_{i=1}^k (-1)^{i+1} a_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$. *We have*

$$\int_{\partial M} \omega = \int_{\partial M} \vec{V} \cdot \vec{n} d\rho^{\text{eucl}}$$

where $\vec{n}$ is the outward pointing unit normal vector field to ∂M.

Example 20.3.4 (Divergence theorem). Let $\omega = a dx_2 \wedge dx_3 - b dx_1 \wedge dx_3 + c dx_1 \wedge dx_2$ be a 2-form on $\mathbb{R}^3$ and $M \subset \mathbb{R}^3$ a codimension 0 submanifold with boundary ∂M with induced orientation from the standard orientation on $\mathbb{R}^3$. Then Stokes' theorem says that

$$\int_M d\omega = \int_{\partial M} \omega.$$

Using the above two lemma's, we get

$$\int_M \operatorname{div}(\vec{V}) d\rho^{\text{eucl}} = \int_{\partial M} \vec{V} \cdot \vec{n} d\rho^{\text{eucl}},$$

the classical statement of *Gauss' divergence theorem* [DK04b, Theorem 7.8.5].

Chapter 21

De Rham cohomology

Today we introduce de Rham cohomology, a construction which we will study for the next couple of lectures and is one of the basic constructions of algebraic topology. It appears in Section 4.§6 of [GP10], but I also recommend you take a look at the beginning of [BT82].

21.1 De Rham cohomology

21.1.1 Motivation from integration

Recall that Stokes' theorem says that for oriented k-dimensional differentiable manifolds M and compactly-supported $(k-1)$-forms ω on M, we have

$$\int_M d\omega = \int_{\partial M} \omega.$$

Thus if M has no boundary, $\int_M d\omega = 0$ for any ω. This means that when computing $\int_M \omega$, its values only depend on ω up to addition by $d\nu$. That is, the possible values that can be obtained when integrating a k-form over a k-dimensional compact oriented manifold M depend only on $\Omega^k(M)/d\Omega^{k-1}(M)$.

One could ask a similar question about integrals over p-dimensional oriented manifolds mapping to M: if X is such a manifold and $f\colon X \to M$ is a smooth map, we are interested in the integral $\int_X f^*\omega$. The argument above tells you that these integrals only depend on $\Omega^p(M)/d\Omega^{p-1}(M)$: we take p-forms modulo those that are exterior derivatives of $(p-1)$-forms. A p-form of the latter type is said to be *exact*.

It seems reasonable to restrict to those p-forms ω with the property if $\int_X f^*\omega$ only depends on the homotopy class of X. As discussed at the end of the previous lecture, from Stokes' theorem applied to $\int_{X\times[0,1]} H^*d\omega$ with $H\colon X \times [0,1] \to M$ a homotopy from f_0 to f_1, it follows that $\int_X f_0^*\omega = \int_X f_1^*\omega$ if $d\omega = 0$, as then $0 = \int_W H^*d\omega = \int_X f_1^*\omega - \int_X f_0^*\omega$. If $d\omega = 0$ then ω is said to be *closed*. Observe that when ω is a k-form then $d\omega = 0$ for degree reasons, so any top form is closed.

21.1.2 De Rham cohomology

The previous discussion tells us that the following groups can be interpreted as encoding "all possible values of homotopy-invariant integrals over manifolds mapping to M." However, you should not take this to be the only motivation. As you will soon see, de Rham cohomology is a powerful invariant of smooth manifolds and smooth maps between them.

Definition 21.1.1. Let M be a manifold. The *de Rham cohomology groups* $H^*(M)$ are given by

$$H^p(M) := \frac{\ker(d\colon \Omega^p(M) \to \Omega^{p+1}(M))}{\operatorname{im}(d\colon \Omega^{p-1}(M) \to \Omega^p(M))} = \frac{\text{closed } p\text{-forms}}{\text{exact } p\text{-forms}}.$$

The elements of $H^*(M)$ are called *cohomology classes* and, as indicated, are represented by closed forms up to exact forms; any two forms which differ by an exact form are said to be *cohomologous*.

21.1.3 First properties of de Rham cohomology

Let us take a closer look at de Rham cohomology. Since $\Omega^p(M)$ are $\mathbb{R}$-vector spaces, so are the cohomology groups $H^p(M)$. We will soon see these $\mathbb{R}$-vector spaces contain a lot of interesting topological information about M. Before going into the properties that allow us to extract this information, we do a few basic examples to get some initial intuition for de Rham cohomology:

Example 21.1.2 (Vanishing above dimension). If M has dimension k, there are no p-forms for $p > k$ and hence $H^p(M)$ vanishes for $p > k$.

Example 21.1.3 (H^0). For $p = 0$, our definition gives that $H^0(M) = \{f\colon M \to \mathbb{R} \mid df = 0\}$. The condition $df = 0$ means that f is locally constant. Thus these functions have to be constant on each component of M, and letting $\pi_0(M)$ denote the set of components of M we get that

$$H^0(M) = \mathbb{R}^{\pi_0(M)},$$

the vector space of $\mathbb{R}$-valued functions on the set $\pi_0(M)$.

Example 21.1.4 (Disjoint unions). Suppose that M is a disjoint union of M_i. Then a p-form ω on M is a just a collection of p-forms ω_i on each of the M_i. Then ω is closed if and only if each ω_i is, and exact if and only if each ω_i is. We conclude that

$$H^*(M) \cong \prod_i H^*(M_i).$$

However, in practice M has finitely many components and the direct product is finite. In this case the direct product may be replaced by the more familiar direct sum.

Recall that we have defined a wedge product on differential forms, and this has the property that if $\omega \in \Omega^p(M)$ and $\nu \in \Omega^q(M)$ then $d(\omega \wedge \nu) = d(\omega) \wedge \nu + (-1)^p \omega \wedge d(\nu)$.

Lemma 21.1.5. *The wedge product induces a graded-commutative product on $H^*(M)$. That is, $H^*(M)$ is a graded-commutative $\mathbb{R}$-algebra.*

Proof. Let $\omega \in \Omega^p(M)$ and $\nu \in \Omega^q(M)$ represent cohomology classes. Then in particular $d\omega = 0$ and $d\nu = 0$, and we see that

$$d(\omega \wedge \nu) = d(\omega) \wedge \nu + (-1)^p \omega \wedge d(\nu) = 0 + 0 = 0.$$

Thus $\omega \wedge \nu$ represents a cohomology class. This is independent of the choice of representatives, because if $\omega - \omega' = d\alpha$, then

$$\omega \wedge \nu - \omega' \wedge \nu = d(\alpha) \wedge \nu = d(\alpha \wedge \nu)$$

and similarly in the second entry.

The properties of this induced product—unitality, associativity, and graded-commutativity—follow from those of the wedge product. □

Example 21.1.6. The unit of the wedge product is the element of $H^0(M)$ represent by the constant function $M \to \mathbb{R}$ with value 1.

21.1.4 Cohomology as a functor

Recall that we can pull back differential forms along any smooth map: given $g\colon M \to N$ we get $g^*\colon \Omega^*(N) \to \Omega^*(M)$.

Lemma 21.1.7. *The homomorphism g^* induces a homomorphism of graded-commutative $\mathbb{R}$-algebras $g^*\colon H^*(N) \to H^*(M)$ which satisfies $(f \circ g)^* = g^* \circ f^*$ and $(\mathrm{id})^* = \mathrm{id}$.*

Proof. We use that d commutes with g^*, so g^* must preserve the kernel and image of d. Let's check this is in detail for kernels: if $\omega \in \Omega^p(N)$ satisfies $d\omega = 0$, then $g^*\omega \in \Omega^p(M)$ satisfies

$$d(g^*\omega) = g^*(d\omega) = g^*0 = 0.$$

The properties of pullback on cohomology follow from the corresponding properties of pullback on forms. □

It is appropriate at this point to mention the foundational framework used in algebraic topology: *category theory* [Rie16]. A *category* C consists of a collection of object $\mathrm{ob}(\mathsf{C})$ and a collection of morphisms $\mathrm{mor}(\mathsf{C})$. Each of these morphisms has a source and a target, and two morphisms f and g can be composed to $g \circ f$ if the target of the f is the source of g. This composition operations is associative, and every object has an identity morphism which serves as a two-sided unit for composition.

The standard way to picture a category is a collection of dots (objects) and arrows between them (morphisms). One instance of such graphic representations are the commutative diagrams we have been using (in the 40s people wrote down the formulas, and it was difficult to parse statements).

Example 21.1.8. The category of Top of topological spaces has objects given by topological spaces, and morphisms given by continuous maps.

Example 21.1.9. The category Mfd of differentiable manifolds has objects given by differentiable manifolds, and morphisms given by smooth maps.

Example 21.1.10. The category $\mathsf{GrAlg}_\mathbb{R}$ of graded-commutative $\mathbb{R}$-algebras has objects given by graded $\mathbb{R}$-vector spaces with a graded-commutative product, and morphisms given by grading-preserving homomorphisms.

An important application of categories is to express naturality of various constructions. For example, that they are compatible with composition is expressed through the notion of a functor. A *functor* $F\colon \mathsf{C} \to \mathsf{D}$ is a pair of assignments $\mathrm{ob}(F)\colon \mathrm{ob}(\mathsf{C}) \to \mathrm{ob}(\mathsf{D})$ and $\mathrm{mor}(F)\colon \mathrm{mor}(\mathsf{C}) \to \mathrm{mor}(\mathsf{D})$, compatible with source, target, identity and composition. The former two mean that if f is a morphism from C to C', then $F(f)$ is a morphism from $F(C) \to F(C')$, and the latter two mean that $F(\mathrm{id}) = \mathrm{id}$ and $F(f \circ g) = F(f) \circ F(g)$.

Example 21.1.11. There is a forgetful functor $U\colon \mathsf{Mfd} \to \mathsf{Top}$ sending each differentiable manifold to its underlying topological spaces, and regarding each smooth map as a continuous map.

It is not the case that cohomology is a functor $H^*\colon \mathsf{Mfd} \to \mathsf{GrAlg}_\mathbb{R}$; it would need to satisfy $(f \circ g)_* = f_* \circ g_*$ but instead we have $(f \circ g)_* = g_* \circ f_*$. This is no problem, as we can formally change the direction of morphisms in Mfd by taking the *opposite category*: $\mathsf{Mfd}^{\mathrm{op}}$ has the same objects and morphisms, but source and target are reversed. Then Lemma 21.1.7 says that de Rham cohomology is a functor

$$H^*\colon \mathsf{Mfd}^{\mathrm{op}} \longrightarrow \mathsf{GrAlg}_\mathbb{R}.$$

As an application of this, we make the following observation, which we will strengthen in the next lecture:

Lemma 21.1.12. *If $g\colon M \to N$ is a diffeomorphism then $g^*\colon H^*(N) \to H^*(M)$ is an isomorphism.*

Proof. The inverse $g^{-1}\colon N \to M$ induces a homomorphism $(g^{-1})^*\colon H^*(M) \to H^*(N)$. The fact that cohomology is a functor tells us that this satisfies $g^* \circ (g^{-1})^* = (g^{-1} \circ g)^* = (\mathrm{id})^* = \mathrm{id}$ and similarly for the other composition. □

21.2 First examples

Let us start with a first few computations in de Rham cohomology, before we develop the techniques that allow us to systematically compute the cohomology of many smooth manifolds.

21.2.1 The real line

We already know that $H^0(\mathbb{R}) \cong \mathbb{R}$ by Example 21.1.3 and that $H^*(\mathbb{R}) = 0$ for $* > 1$ by Example 21.1.2, so the only remaining unknown cohomology group is $H^1(\mathbb{R})$. Any element in it is represented by $\omega \in \Omega^1(\mathbb{R})$ (satisfying $d\omega = 0$, but this is true for any such ω for degree reasons).

Lemma 21.2.1. $H^1(\mathbb{R}) = 0$.

Proof. We need to find an f such that $\omega = df$. Let us write $\omega = a(x)dx$ with $a\colon \mathbb{R} \to \mathbb{R}$ a smooth function, then

$$\begin{aligned} f\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \int_0^x a(y)dy \end{aligned}$$

satisfies $df(x) = \frac{\partial f}{\partial x}dx = a(x)dx$. That is, every closed 1-form is exact. □

In the next lecture we will prove the Poincaré lemma, which says that for all $n \geqslant 0$

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

21.2.2 The circle

For the circle S^1, we are in a somewhat similar situation as for the real line: $H^0(S^1) = \mathbb{R}$ and $H^*(S^1) = 0$ for $* > 1$, so only $H^1(S^1)$ remains unknown.

Lemma 21.2.2. $H^1(S^1) = \mathbb{R}$.

First proof. Let us write $\omega = a(\theta)d\theta$ with $a\colon S^1 \to \mathbb{R}$ a smooth function, then the argument for the real line compels us to look at the function

$$\begin{aligned} f\colon [0, 2\pi] &\longrightarrow \mathbb{R} \\ \theta &\longmapsto \int_0^\theta a(e^{i\phi})d\phi. \end{aligned}$$

This gives a smooth function on S^1 if and only if $f(0) = f(2\pi)$.

This gives an obstruction to implementing to proving that $H^1(S^1)$ vanishes along the lines of the proof for $\mathbb{R}$. But instead of giving up, we should take advantage of this and use the obstruction to define an invariant. That is, we can attempt to construct a linear functional on $H^1(\mathbb{R})$ by taking

$$\begin{aligned} w\colon H^1(S^1) &\longrightarrow \mathbb{R} \\ \omega = a(\theta)d\theta &\longmapsto \int_0^{2\pi} a(e^{i\phi})d\phi. \end{aligned}$$

To check this is well-defined, we must verify it is independent of the representative ω of the cohomology class $[\omega] \in H^1(\mathbb{R})$. As w is linear in ω, so it suffices to show that $w(\omega) = 0$ if $\omega = df$ for a smooth function $f\colon S^1 \to \mathbb{R}$. This is true because the integral is equal to $f(2\pi) - f(0) = 0$ by the fundamental theorem of calculus.

If $w(\omega) = 0$ then $f(0) = f(2\pi)$, and gives a smooth function $S^1 \to \mathbb{R}$ which we can use to show that $\omega = df$ like we did for $\mathbb{R}$. Hence the result follows once we show that w is surjective. Since w is linear it suffices to prove that it takes a single non-zero value, and when we evaluate on the 1-form $\omega = d\theta$ we get $w(d\theta) = 2\pi$. □

Let's give an alternative proof, which we will later generalize to the Mayer–Vietoris exact sequence for cohomology.

Proof. Let $U, V \subset S^1$ be an open cover by two open intervals and consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(S^1) & \xrightarrow{i_0} & \Omega^0(U) \oplus \Omega^0(V) & \xrightarrow{j_0} & \Omega^0(U \cap V) & \longrightarrow & 0 \\
 & & \Big\downarrow d & & \Big\downarrow d & & \Big\downarrow d & & \\
0 & \longrightarrow & \Omega^1(S^1) & \xrightarrow{i_1} & \Omega^1(U) \oplus \Omega^1(V) & \xrightarrow{j_1} & \Omega^1(U \cap V) & \longrightarrow & 0
\end{array}$$

The left horizontal maps are induced by restrictions,

$$\begin{aligned} i_0 \colon \Omega^0(S^1) &\longrightarrow \Omega^0(U) \oplus \Omega^0(V) \\ f &\longmapsto (f|_U, f|_V), \end{aligned}$$

and similarly for i_1. The right horizontal maps are the difference of the restrictions,

$$\begin{aligned} j_0 \colon \Omega^0(U) \oplus \Omega^0(V) &\longrightarrow \Omega^0(U \cap V) \\ (f, g) &\longmapsto f|_{U\cap V} - g|_{U \cap V}, \end{aligned}$$

and similarly for j_1.

We start with a 1-form $\omega \in \Omega^1(S^1)$ representing a cohomology class $[\omega]$, and consider $i(\omega) = (\omega|_U, \omega|_V) \in \Omega^1(U) \oplus \Omega^1(V)$. Since ω was closed, so are both these restrictions. Since $H^1(U)$ and $H^1(V)$ vanish because both U and V are diffeomorphic to $\mathbb{R}$, both are exact. This gives us functions $(\lambda_U, \lambda_V) \in \Omega^0(U) \oplus \Omega^0(V)$.

Let us investigate to what extent

$$j_0(\lambda_U, \lambda_V) = \lambda_U|_{U\cap V} - \lambda_V|_{U\cap V} \in \Omega^0(U \cap V)$$

depends on the choices we made. We made two:

(a) the functions (λ_U, λ_V),

(b) a representative ω of $[\omega]$.

For (a), the functions λ_U and λ_V are unique up to the addition of constant functions, i.e. closed 0-forms. Adding a constant to one of these changes $j_0(\lambda_U, \lambda_V)$ by a constant.

For (b), a different representative is given by $\omega + df$ with $f \in \Omega^0(S^1)$, and picking this instead leads to replacing λ_U by $\Lambda_U + f|_U$ and λ_V by $\lambda_V + f_| V$, up to constants. When we take $j_0(\lambda_U + f|_U, \lambda_V + f|_V)$ the terms $f|_{U\cap V}$ cancel out and we get $j_0(\lambda_U, \lambda_V)$

The conclusion is that the smooth function $j_0(\lambda_U, \lambda_V) \in \Omega^0(U \cap V)$ is independent of the choice of representative ω, and depends on λ_U and λ_V only up to a constant. Since both $\omega|_U$ and $\omega|_V$ are equal to $\omega|_{U\cap V}$ on $U \cap V$ and the exterior derivative is linear, we see that

$$d(j(\lambda_U, \lambda_V)) = d(\lambda_U|_{U\cap V} - \lambda_V|_{U\cap V}) = \omega|_{U\cap V} - \omega|_{U\cap V} = 0,$$

i.e. $j(\lambda_U, \lambda_V)$ is closed, or equivalently a locally constant function on $U \cap V$. Under the identification as Examples 21.1.4 and 21.1.3, it represents an element

$$(a_0, a_1) \in H^0(U \cap V) \cong \mathbb{R}^2.$$

That is well-defined up the addition of a constant, means that we may replace (a_0, a_1) by $(a_0 + c, a_1 + c)$. The elements of the form (c, c) are exactly those in the image of $H^0(U) \oplus H^0(V)$ under j_0.

From this description, it follows that $a_1 - a_0 \in \mathbb{R}$ is independent of the choice of λ_U and λ_V; an invariant of the original cohomology class $[\omega]$. Thus we have constructed a map

$$\overline{w}\colon H^1(S^1) \longrightarrow \frac{H^0(U \cap V)}{\operatorname{im}(j\colon H^0(U) \oplus H^0(V) \to H^0(U \cap V))} \cong \mathbb{R}.$$

Suppose now that $\overline{w}(\omega) = 0$. Then $a_0 = a_1$ and this means that though the functions λ_U and λ_V needs not be equal on $U \cap V$, the difference $\lambda_U|_{U\cap V} - \lambda_V|_{U\cap V}$ is constant. We can thus replace λ_U by $\lambda_U - a_0$ to get that $\lambda_U|_{U\cap V} = \lambda_V|_{U\cap V}$. Hence we can glue them to obtain a function λ on S^1, which by construction satisfies $d\lambda = \omega$.

This shows that $H^1(S^1)$ is isomorphic to the image of $\overline{w}$. To see that it is surjective, as before we can evaluate on $d\theta$. $\square$

Remark 21.2.3. The construction of $\overline{w}$ depends on a choice of isomorphism of the codomain with $\mathbb{R}$. You can pick this such that $w = \overline{w}$.

The previous proof amounts to the following: the maps i and j induce maps on cohomology and using partitions of unity one can produce a diagonal map to get the following diagram:

$$\begin{tikzcd}
H^1(S^1) \arrow[r] & H^1(U) \oplus H^1(V) \arrow[r] & H^1(U \cap V) \\
H^0(S^1) \arrow[r] & H^0(U) \oplus H^0(V) \arrow[r] & H^0(U \cap V) \arrow[ull, hook]
\end{tikzcd}$$

This diagram has the special property that it is *exact*: the kernel of each map is the image of the previous one. Filling in what we already know, we get

$$\begin{tikzcd}
H^1(S^1) \arrow[r] & H^1(U) \oplus H^1(V) = 0 \arrow[r] & H^1(U \cap V) = 0 \\
H^0(S^1) = \mathbb{R} \arrow[r] & H^0(U) \oplus H^0(V) = \mathbb{R}^2 \arrow[r, "(*)"] & H^0(U \cap V) = \mathbb{R}^2 \arrow[ull, hook]
\end{tikzcd}$$

with starred map given by $(a, b) \longmapsto (a - b, a - b)$. This proves that $H^1(S^1)$, the kernel of the map to $H^1(U) \oplus H^1(V) = 0$, is the image of the map $H^0(U \cap V) = \mathbb{R}^2 \to H^1(S^1)$ whose kernel is exactly the 1-dimensional subspace spanned by $e_1 + e_2$. Hence $H^1(S^1) \cong \mathbb{R}$.

What the above proof does, is construct explicitly the identification

$$\begin{array}{c} H^1(S^1) = \ker(H^1(S^1) \to H^1(U) \oplus H^1(V)) \\ \overline{w}\Big\downarrow\cong \\ \dfrac{H^0(U \cap V)}{\operatorname{im}(j\colon H^0(U) \oplus H^0(V) \to H^0(U \cap V))} \cong \mathbb{R}. \end{array}$$

21.3 Problems

Problem 48. Verify that $\overline{\omega}(d\theta) \neq 0$.

Problem 49 (Compactly-supported cohomology)**.** Recall that $\Omega_c^p(M)$ denotes the compactly-supported p-forms. Since the exterior derivative d preserves the condition that forms have compact support, there is also a compactly-supported variation of de Rham cohomology which is occasionally useful:

Definition 21.3.1. The *compactly-supported de Rham cohomology groups* $H_c^*(M)$ are given by

$$H_c^p(M) := \frac{\ker(d\colon \Omega_c^p(M) \to \Omega_c^{p+1}(M))}{\operatorname{im}(d\colon \Omega_c^{p-1}(M) \to \Omega_c^p(M))}.$$

(a) Compute $H_c^*(\mathbb{R})$.

(b) Compute $H_c^*(S^1)$. (Hint: this should require no work.)

Problem 50 (Extension by zero)**.** Prove that if $i\colon U \to M$ is the inclusion of an open subset, the extension of forms by zero induces a map

$$H_c^*(U) \longrightarrow H_c^*(M)$$

on compactly-supported cohomology.

Problem 51 (An infinitely-punctured plane)**.** Prove that $H^1(\mathbb{C}\backslash\mathbb{Z})$ is not finite-dimensional.

Problem 52 (Transfer maps)**.** Let M be a smooth manifold with a smooth free action of a finite group G, with $a_g\colon M \to M$ denoting the action of the elements $g \in G$. Recall that M/G can be given the structure of a smooth manifold such that quotient map $q\colon M \to M/G$ is a local diffeomorphism.

(a) Let $\Omega^*(M)^G \subset \Omega^*(M)$ be the subspace given by those differential forms that satisfy $(a_g)^*\omega = \omega$ for all $g \in G$; the *invariant forms.* Prove that $\Omega^*(M)^G$ is a cochain complex with differential given by exterior derivative, and prove that it is isomorphic as a cochain complex to $\Omega^*(M/G)$.

(b) Show that the map

$$\Omega^*(M) \ni \omega \longmapsto \frac{1}{|G|}\sum_{g\in G}(a_g)^*\omega$$

gives a map of cochain complexes $\Omega^*(M) \to \Omega^*(M)^G$. The induced map on cohomology is called the *transfer map.*

(c) What is the composition $\Omega^*(M)^G \to \Omega^*(M) \to \Omega^*(M)^G$? Show that the pullback map $q^*\colon H^*(M/G) \to H^*(M)$ is injective.

(d) Let S^3/I^* be the Poincare homology sphere. Prove that $H^*(S^3/I^*) \cong H^*(S^3)$.

(e) Explain how to obtain $H^*(\mathbb{R}P^3)$ from the above results without doing any additional computation.

Chapter 22

The Poincaré lemma

Last lecture we introduced de Rham cohomology, and today we prove the Poincaré lemma. This is proven in [GP10, Section 4.§6] and [BT82, Section 4].

22.1 The Poincaré lemma

The Poincaré lemma computes the cohomology of $\mathbb{R}^n$. It is the backbone of all further computations of cohomology groups.

22.1.1 The Poincaré lemma on $\mathbb{R}^n$

In the previous chapter we defined a functor

$$H^*(-)\colon \mathsf{Mfd}^{\mathrm{op}} \longrightarrow \mathsf{GrAlg}_{\mathbb{R}},$$

sending a manifold M to the graded-commutative $\mathbb{R}$-algebra $H^*(M)$ of *de Rham cohomology groups*

$$H^p(M) := \frac{\ker(d\colon \Omega^p(M) \to \Omega^{p+1}(M))}{\operatorname{im}(d\colon \Omega^{p-1}(M) \to \Omega^p(M))} = \frac{\text{closed } p\text{-forms}}{\text{exact } p\text{-forms}}.$$

It sends a smooth map $f\colon M \to N$ to the homomorphism $f^*\colon H^*(N) \to H^*(M)$ induced by pullback of differential forms.

We also computed $H^*(S^1)$ and $H^*(\mathbb{R})$, obtaining the following computation in the latter case

$$H^*(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Our immediate goal is to extend this computation to $\mathbb{R}^n$, by induction over n. A more precise statement uses the projection

$$\begin{aligned} \pi\colon \mathbb{R}^{n-1} \times \mathbb{R} &\longrightarrow \mathbb{R}^{n-1} \\ (x,t) &\longmapsto x, \end{aligned}$$

as well as the map s_{t_0} for $t_0 \in \mathbb{R}$,

$$\begin{aligned} s_{t_0}\colon \mathbb{R}^{n-1} &\longrightarrow \mathbb{R}^{n-1} \times \mathbb{R} \\ x &\longmapsto (x, t_0). \end{aligned}$$

These satisfy $\pi \circ s_{t_0} = \mathrm{id}_{\mathbb{R}^{n-1}}$. It is of course not true that $s_{t_0} \circ \pi$ is the identity; it is given by $(x, t) \mapsto (x, t_0)$. Nonetheless, on cohomology we have:

Theorem 22.1.1 (Poincaré lemma)**.** *For each $t_0 \in \mathbb{R}$, the map $s_{t_0}^* \colon H^*(\mathbb{R}^{n-1} \times \mathbb{R}) \to H^*(\mathbb{R}^{n-1})$ is an isomorphism with inverse π^*.*

By induction over n, starting with the case $n = 0$ where $\mathbb{R}^0 = *$ (so in particular, we reprove the case $n = 1$), one can use this to prove:

Corollary 22.1.2. *We have that*

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 22.1.1. Let us shorten s_{t_0} to s for the sake of brevity. Above we observed that $\pi \circ s = \mathrm{id}$ so that we get $s^* \circ \pi^* = \mathrm{id}^* = \mathrm{id}$ on $H^*(\mathbb{R}^{n-1})$. It is of course not true that $s \circ \pi = \mathrm{id}$. However, we will still prove that the induced maps on de Rham cohomology satisfy $\pi^* \circ s^* = \mathrm{id}$ on $H^*(\mathbb{R}^{n-1} \times \mathbb{R})$.

To do so, we will prove that there is a map $K \colon \Omega^*(\mathbb{R}^{n-1} \times \mathbb{R}) \to \Omega^{*-1}(\mathbb{R}^{n-1} \times \mathbb{R})$ satisfying

$$\mathrm{id} - \pi^* \circ s^* = (-1)^{p-1}(dK - Kd). \tag{22.1}$$

This tells us that on closed forms in $\Omega^*(\mathbb{R}^{n-1} \times \mathbb{R})$, id and $\pi^* \circ s^*$ differ by an exact form, and hence give the same cohomology class.

To define K, we use coordinates $(x_1, \ldots, x_{n-1}, t)$ on $\mathbb{R}^{n-1} \times \mathbb{R}$ observe that every p-form in $\mathbb{R} \times \mathbb{R}^{n-1}$ can be uniquely written as a linear combination of p-forms of the following forms:

(i) $a_I(x, t) dx_I$ with $|I| = p$,

(ii) $a_J(x, t) dx_J \wedge dt$ and $|J| = p - 1$.

The map K will be linear, so it is uniquely determined by demanding it satisfies

$$K(a_I(x,t)dx_I) = 0 \qquad \text{and} \qquad K(a_J(x,t)dx_J \wedge dt)(x,t) = \left(\int_{t_0}^t a_J(x,s)ds\right) dx_J.$$

We verify that $(-1)^{p-1}(dK - Kd) = \mathrm{id} - \pi^* \circ s^*$. First we do so for forms of type (i). On such forms we have that $\pi^* \circ s^*(a_I(x,t)dx_I) = a_I(x, t_0)dx_I$, so that

$$(\mathrm{id} - \pi^* \circ s^*)(a_I(x,t)dx_I) = (a(x,t) - a(x,t_0))dx_I.$$

On the other hand, we have at (x, t)

$$
\begin{aligned}
(-1)^{p-1}&(dK - Kd)(a_I(x,t)dx_I) \\
&= (-1)^p K\left(\frac{\partial a_I(x,t)}{\partial t}dt \wedge dx_I + \sum_{i=1}^{n-1}\frac{\partial a_I(x,t)}{\partial x_i}dx_i \wedge dx_I\right) \\
&= (-1)^p K\left(\frac{\partial a_I(x,t)}{\partial t}dt \wedge dx_I\right) \\
&= K\left(\frac{\partial a_I(x,t)}{\partial t}dx_I \wedge dt\right) \\
&= \left(\int_{t_0}^{t}\frac{\partial a_I(x,s)}{\partial t}ds\right)dx_I \\
&= (a_I(x,t) - a_I(x,t_0))dx_I.
\end{aligned}
$$

We conclude that (22.1) holds on forms of type (i).

For forms of type (ii), we observe that $\pi^* \circ s^*(a_J(x,t)dx_J \wedge dt) = 0$ because $s^*dt = 0$, so that

$$(\mathrm{id} - \pi^* \circ s^*)(a_J(x,t)dx_J \wedge dt) = a_J(x,t)dx_J \wedge dt.$$

On the other hand, for $(-1)^{p-1}(dK - Kd)(a_J(x,t)dx_J \wedge dt)$ we do two separate computations

$$
\begin{aligned}
Kd(a_J(x,t)dx_J \wedge dt) &= K\left(\frac{\partial a_J(x,t)}{\partial t}dt \wedge dx_J \wedge dt + \sum_{i=1}^{n-1}\frac{\partial a_J(x,t)}{\partial x_i}dx_i \wedge dx_J \wedge dt\right) \\
&= \sum_{i=1}^{n-1} K\left(\frac{\partial a_J(x,t)}{\partial x_i}dx_i \wedge dx_J \wedge dt\right) \\
&= \sum_{i=1}^{n-1}\left(\int_{t_0}^{t}\frac{\partial a_J(x,s)}{\partial x_i}ds\right)dx_i \wedge dx_I.
\end{aligned}
$$

$$
\begin{aligned}
dK(a_J(x,t)dx_J \wedge dt) &= d\left(\left(\int_{t_0}^{t} a_J(x,)ds\right)dx_J\right) \\
&= \frac{\partial \int_{t_0}^{t} a_J(s,x)ds}{\partial t}dt \wedge dx_J + \sum_{i=1}^{n-1}\frac{\partial \int_{t_0}^{t} a_J(x,s)ds}{\partial x_i}dx_i \wedge dx_J \\
&= a_J(x,t)dt \wedge dx_J + \sum_{i=1}^{n-1}\left(\int_{t_0}^{t}\frac{\partial a_J(x,s)}{\partial x_i}ds\right)dx_i \wedge dx_J \\
&= (-1)^{p-1}a_J(x,t)dx_J \wedge dt + Kd(a_J(x,t)dx_J \wedge dt).
\end{aligned}
$$

Hence $(-1)^{p-1}(dK - Kd)(a_J(x,t)dx_J \wedge dt) = a_J(x,t)dx_J \wedge dt$, so (22.1) also holds on forms of type (ii). □

Remark 22.1.3. A map such as K is called a *cochain homotopy*, and (22.1) says that id and $\pi^* \circ s^*$ are *cochain homotopic*.

22.1.2 The Poincaré lemma on manifolds

The proof of Theorem 22.1.1 goes through without any modification when we replace $\mathbb{R}^{n-1}$ by any open subset $U \subset \mathbb{R}^{n-1}$. We can do even better:

Corollary 22.1.4. *For each $t_0 \in \mathbb{R}$ and smooth manifold M, the map $s_{t_0}^*\colon H^*(M \times \mathbb{R}) \to H^*(M)$ is an isomorphism with inverse π^*.*

Proof. We can describe the types (i) and (ii) in a coordinate-invariant manner: (i) are those of the form $f(x,t)\pi^*\omega$, (ii) are those of the form $f(x,t)\pi^*(\omega) \wedge dt$. Since the cotangent bundle of $M \times \mathbb{R}$ is isomorphic to $\pi^*(T^*M) \oplus \epsilon$, every form on $M \times \mathbb{R}$ can be written uniquely as a linear combination of forms of type (i) and (ii). Now the proof given above goes through with the modification that $\mathbb{R}^{n-1}$ is replaced by M. ☐

Example 22.1.5. The previous corollary proves that open annulus $\mathbb{A}$ has the same cohomology as S^1, as it is diffeomorphic to $S^1 \times \mathbb{R}$.

22.2 Homotopy invariance

22.2.1 Homotopy invariance for de Rham cohomology

Corollary 22.1.4 says that π^* has as its inverse $s_{t_0}^*$ *for any* t_0. Since inverses are unique, this means that the maps $s_{t_0}^*\colon H^*(M \times \mathbb{R}) \to H^*(M)$ are all equal. Recall that $f_0, f_1\colon M \to N$ are homotopic if there is a map $H\colon M \times \mathbb{R} \to N$ such that $H|_{M\times\{0\}} = f_0$ and $H|_{M\times\{1\}} = f_1$, then this has the following important consequence.

Theorem 22.2.1 (Homotopy invariance)**.** *If $f_0, f_1\colon M \to N$ are homotopic smooth maps, then $f_0^* = f_1^*\colon H^*(N) \to H^*(M)$.*

Proof. We can find a homotopy of the form $H\colon M \times \mathbb{R} \to N$. We can then factor f_i, $i = 0, 1$ as

$$\begin{tikzcd} M \arrow[r, "s_i"] \arrow[rr, bend left, "f_i"] & M \times \mathbb{R} \arrow[r, "H"] & N, \end{tikzcd}$$

and obtain equations

$$f_0^* = s_0^* \circ H^* = s_1^* \circ H^* = f_1^*.$$ ☐

Recall that we proved that every diffeomorphism induces an isomorphism on cohomology, that is, every smooth map with an inverse does. It actually suffices that f has an inverse *up to homotopy.*

Corollary 22.2.2. *If $f\colon M \to N$ is a homotopy equivalence, then $f^*\colon H^*(N) \to H^*(M)$ is an isomorphism.*

Example 22.2.3. If M is a Moebius strip, then the inclusion $S^1 \hookrightarrow M$ is a homotopy equivalence (the homotopy inverse is the bundle projection). Thus $H^*(M) \cong H^*(S^1)$.

More generally, if E is the total space of a smooth vector bundle over M, then $H^*(E) \cong H^*(M)$. This is a generalization of Corollary 22.1.4; that corollary can be interpreted as saying that the total spaces of 1-dimensional trivial bundles have the same cohomology as their 0-section.

Remark 22.2.4. At this point you can extend cohomology to a large class of spaces in a rather artificial manner. For example, if K is built by gluing together finitely many simplices (i.e. vertices, edges, triangles, tetrahedra, etc.), it has an embedding into a sufficiently large Euclidean space with a small open neighbourhood U that is unique up to homotopy equivalence. Thus setting $H^*(K) := H^*(U)$ gives a well-defined notion of cohomology for such spaces K. However, algebraic topology provides an elegant definition of cohomology (with real coefficients) for any topological space. It is then a theorem that this coincides with de Rham cohomology when evaluated on a manifold; *de Rham's theorem.*

22.2.2 Applications

Contractible manifolds

Recall that there exists contractible manifolds M which are not homeomorphic to Euclidean space, such as the Whitehead manifold. Nonetheless, the homotopy invariance of de Rham cohomology implies these have the same cohomology as Euclidean spaces:

$$H^*(M) = \begin{cases} \mathbb{R} & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The interior of a manifold with boundary

If M is a manifold with boundary ∂M, then we saw that there is an interior collar $\rho \colon \partial M \times [0,1) \hookrightarrow M$.

Lemma 22.2.5. *The inclusion* $\mathrm{int}(M) \hookrightarrow M$ *is a homotopy equivalence.*

Proof. The homotopy inverse h is given in terms of the collar as

$$h(p) = \begin{cases} p & \text{if } p \notin \mathrm{im}(\rho), \\ \rho(x, \eta(t)) & \text{if } p = \rho(x,t), \end{cases}$$

where $\eta \colon [0,1] \to [0,1]$ is an embedding that is the identity near 1 and has image given by $[1/2, 1]$. Intuitively, we push the manifold into itself a bit using the collar. We leave it to reader to convince themselves that $i \circ g$ and $g \circ i$ are homotopic to the identity. $\square$

The homotopy invariance of cohomology then gives us:

Corollary 22.2.6. $H^*(M) \cong H^*(\mathrm{int}(M))$.

Brouwer fixed point theorem

Observe that $\mathrm{int}(D^2)$ is diffeomorphic to $\mathbb{R}^2$ by $x \mapsto x/(1+||x||^2)$. By the previous corollary we obtain $H^1(D^2) \cong H^1(\mathbb{R}^2) = 0$. Let us use this to give another proof of the Brouwer fixed point theorem for D^2. Recall that this follows from the following "no-retraction" theorem:

Corollary 22.2.7. *There exists no smooth retraction $r\colon \partial D^2 \to D^2$.*

Proof. If such an r did exist, we would have a commutative diagram

$$\begin{tikzcd} \partial D^2 \arrow[rr, "\mathrm{id}"] \arrow[dr, hook, "\mathrm{inc}"'] & & \partial D^2 \\ & D^2 \arrow[ur, "r"'] & \end{tikzcd}$$

and applying the contravariant functor $H^1(-)$ turns this into a commutative diagram

$$\begin{tikzcd} H^1(\partial D^2) \cong \mathbb{R} & & H^1(\partial D^2) \cong \mathbb{R} \arrow[ll, "\mathrm{id}"] \arrow[dl, "r^*"] \\ & H^1(D^2) = 0 \arrow[ul, "\mathrm{inc}^*"] & \end{tikzcd}$$

which is obviously impossible: the identity on $\mathbb{R}$ does not factor over 0. ☐

22.3 Two further tricks

For later use, I will give two further tricks to compute two particular de Rham cohomology groups. For now, the reader should take this as an opportunity to get familiar with de Rham cohomology.

22.3.1 Top degree

Suppose that M is a compact oriented k-dimensional manifold. That M is oriented means that the top exterior power $\Lambda^k T^*M$ has an everywhere non-vanishing section ω. Thus writing ω as $a dx_1 \wedge \cdots \wedge dx_k$ in a chart the function a is always non-vanishing. We intend to integrate this over M. To do so, we must use charts compatible with the orientation; in that case a must be positive. Hence, when computing $\int_M \omega$, we get a finite sum of *non-negative numbers, at least one of which is positive* and hence $\int_M \omega > 0$. Now recall that integration of forms over M gives a linear functional $H^k(M) \to \mathbb{R}$, so that we have just shown the following:

Proposition 22.3.1. *Suppose that M is a compact oriented manifold of dimension k, then* $\dim H^k(M) \geqslant 1$.

We will later prove that its dimension is exactly 1 under the additional hypothesis that M is connected.

Example 22.3.2. For any sphere S^n, we have that $\dim H^n(S^n) \geqslant 1$. Later we will prove it is exactly 1-dimensional.

22.3.2 Degree one

In the homework, you have proven that if $\gamma\colon S^1 \to M$ is a smooth map then $\int_{S^1} \gamma^*\alpha$ has the following properties: (i) if α is exact it is zero, (ii) if α is closed it only depends on the homotopy class of γ. Furthermore, you have seen (iii) given a closed α, if M connected and the integrals $\int_{S^1} \gamma^*\alpha$ vanish for all γ, then α is exact.

If M is connected and we pick a base point $p_0 \in M$, we can define $\pi_1(M, p_0)$ to be the set of based homotopy classes of loops in M; this is the *fundamental group of M at p_0*. Part (i) and (ii) say there is a map

$$\begin{aligned} h\colon \pi_1(M, p_0) &\longrightarrow (H^1(M))^* \\ \gamma &\longmapsto \int_{S^1} \gamma^*\alpha \end{aligned}$$

and part (iii) says that the span of the image of h is all of $(H^1(M))^*$ (at least if it is finite-dimensional, otherwise it is dense). We did not discuss the group structure of $\pi_1(M, p_0)$, but if you know this you will realize h is a homomorphism. It is called the *Hurewicz homomorphism.*

Proposition 22.3.3. *If M is simply-connected, then $H^1(M) = 0$.*

Example 22.3.4. We used Sard's lemma to prove that S^n is simply-connected if $n \geqslant 2$, and hence $H^1(S^n) = 0$.

Remark 22.3.5. The Hurewicz homomorphism factors over $\pi_1(M, p_0)^{ab} \otimes \mathbb{R}$. When M is compact and connected, the resulting homomorphism $\pi_1(M, p_0)^{ab} \otimes \mathbb{R} \to (H^1(M))^*$ is in fact an isomorphism. Thus you can compute $H^1(M)$ knowing the fundamental group.

22.4 Problems

Problem 53 (The Poincaré lemma for compactly-supported cohomology)**.** Read pages 37–39 of [BT82] about the Poincaré lemma for compactly-supported cohomology. This says in particular that

$$H^*_c(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Explain why this shows that compactly-supported cohomology is *not* homotopy-invariant.

Chapter 23

The Mayer–Vietoris theorem

Last lecture we proved the Poincaré lemma, which computes the cohomology of $\mathbb{R}^n$. To exploit that computation, we now prove a "patching theorem" for de Rham cohomology. It is a generalization of the second proof we gave of $H^1(S^1) = \mathbb{R}$. This is proven in Section 4.§6 of [GP10] and Section 2 of [BT82].

23.1 Some homological algebra

Recall that de Rham cohomology of M was constructed from the sequence of $\mathbb{R}$-vector spaces

$$\cdots \longrightarrow \Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \longrightarrow \cdots.$$

by taking the kernel of d modulo the image of d.

This is an example of the cohomology of a cochain complex of $\mathbb{R}$-vector spaces. I will drop the $\mathbb{R}$ from now on. Let me point out that the fact that we're working with vector spaces plays no role in the arguments that follow; we can replace vector spaces by abelian groups, or modules over any ring.

Definition 23.1.1. A *cochain complex* C^* is a collection of vector spaces with linear maps between them

$$\cdots \longrightarrow C^{p-1} \xrightarrow{d} C^p \xrightarrow{d} C^{p+1} \longrightarrow \cdots$$

satisfying $d^2 = 0$. This equation implies $\mathrm{im}(d\colon C^{p-1} \to C^p)$ is a subset of $\ker(d\colon C^p \to C^{p+1})$, and hence it makes sense to define the cohomology groups $H^*(C^*)$ as

$$H^p(C^*) := \frac{\ker(d\colon C^p \to C^{p+1})}{\mathrm{im}(d\colon C^{p-1} \to C^p)}.$$

Definition 23.1.2. A homomorphism of cochain complexes $f\colon B^* \to C^*$ is a collection of linear maps $f_p\colon B^p \to C^p$ such that $df_p = f_{p+1}d$. This condition implies that f induces a map on cohomology.

23.1.1 Short exact sequences of cochain complexes

A *long exact sequence* is a sequence of vector spaces

$$\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow \cdots$$

such that the kernel of each map is the image of the previous one. In other words, it is a cochain complex whose cohomology vanishes at each point.

A *short exact sequence* is a sequence of vector spaces

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

such that the kernel of each map is the image of the previous one. That is, it is just a long exact sequence in which all but three groups vanish. Conretely having a short exact sequence means the following:

- Since the kernel of i is the image of $0 \to A$, i is injective.
- Since the image of j is the kernel of the $C \to 0$, j surjective.
- The kernel of j is the image of i.

Example 23.1.3. Having a short exact sequence is quite useful. For example, suppose you want to compute what a particular vector spaces A is isomorphic to, and you know it fits into a short exact sequence

$$0 \xrightarrow{i} \mathbb{R} \xrightarrow{j} A \xrightarrow{k} \mathbb{R} \xrightarrow{l} 0.$$

Then $\mathbb{R}$ is the kernel of a surjective map $A \to \mathbb{R}$, and thus A must be 2-dimensional.

A short exact sequence of cochain complexes is a sequence of cochain complexes

$$0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$$

such that each sequence

$$0 \longrightarrow A^p \longrightarrow B^p \longrightarrow C^p \longrightarrow 0$$

is a short exact sequence. The following result relates the cohomology groups $H^*(A^*)$, $H^*(B^*)$, and $H^*(C^*)$.

Theorem 23.1.4. *If* $0 \to A^* \to B^* \to C^* \to 0$ *is a short exact sequence of cochain complexes then there exist homomorphisms* $\delta\colon H^p(C^*) \to H^{p+1}(A^*)$ *such that*

$$\cdots \longrightarrow H^p(B^*) \longrightarrow H^p(C^*) \longrightarrow H^{p+1}(A^*) \longrightarrow H^{p+1}(B^*) \longrightarrow \cdots$$

is a long exact sequence.

The homomorphisms δ are called *boundary maps*, and will be constructed explicitly.

Proof. We start with construction of the homomorphism $\delta\colon H^p(C^*) \to H^{p+1}(A^*)$. To do so, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{p+2} & \xrightarrow{i_{p+2}} & B^{p+2} & \xrightarrow{j_{p+2}} & C^{p+2} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^{p+1} & \xrightarrow{i_{p+1}} & B^{p+1} & \xrightarrow{j_{p+1}} & C^{p+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^{p} & \xrightarrow{i_{p}} & B^{p} & \xrightarrow{j_{p}} & C^{p} & \longrightarrow & 0.
\end{array}$$

Let $[x] \in H^p(C^*)$ be represented by $x \in C^p$, then since $j_p\colon B^p \to C^p$ is surjective there exists a lift $y \in B^p$. This satisfies $j_{p+1}(dy) = dj_p(y) = dx = 0$. Thus $dy \in B^{p+1}$ in the kernel of j_{p+1} and hence in the image of i_{p+1}, so there exists a lift $z \in A^{p+1}$.

We want to set $\partial[x] = [z]$. To show that this makes sense, we need to first check that $dz = 0$. Since i_{p+2} is injective, we might as well check that $i_{p+2}(dz) = 0$. But $i_{p+2}(dz) = d(i_{p+1}(z)) = d(dy) = 0$.

Next we need to prove that $[z]$ is independent of the three choices we made:

(a) the choice of representative $x \in C^p$ of $[x]$,

(b) the choice of lift $y \in B^p$ of x, and

(c) the choice of lift $z \in A^{p+1}$ of $d(y)$.

The last of these, (c), in fact involved no choice at all. The element z is unique because i_{p+1} is injective. For (b), any other choice of lift y differs by an element $i_p(w)$, which changes dy to $d(y + i_p(w)) = dy + i_w(dw)$ which has lift to A^{p+1} given by $z + dw$, and hence gives rise to the same cohomology class $[z]$. Finally, for (a), any other representative of x differs by du for $u \in C^{p-1}$. We may lift u to $v \in B^{p-1}$ and then choose to lift of $x + du$ to $y + dv$ (we have already shown that the end result is independent of the choice of lift). Then $d(y + dv) = dy$, so the resulting class $[z]$ is the same as before.

Let us only check exactness at the term $H^p(C^*)$, leaving the other cases for the reader. We need to prove that if $\delta([x]) = 0$ then $[x]$ is in the image of $H^p(B^*)$. Indeed, if $\delta([x]) = 0$ then $z = da$ for some $a \in A^p$. Then $di_p(a) = i_{p+1}(z) = dy$, so $y - i_p(a) \in B^p$ is closed. Furthermore $j_p(y - i_p(a)) = j_p(y) = x$ since $j_p \circ i_p = 0$, so $[x]$ is the image of $[y - i_p(a)]$. □

Remark 23.1.5. A proof as above is hard to read. You should draw the diagram and pencil in were all of the elements discussed live and are mapped. This is called *diagram-chasing.*

23.2 The Mayer–Vietoris theorem

Let M be a manifold, and $U, V \subset M$ be open subsets covering M. Then the maps induced by restriction give rise to a pair of maps

$$\begin{aligned} \Omega^p(M) &\longrightarrow \Omega^p(U) \oplus \Omega^p(V) \\ \omega &\longmapsto (\omega|_U, \omega|_V), \end{aligned}$$

$$\begin{aligned} \Omega^p(U) \oplus \Omega^p(V) &\longrightarrow \Omega^p(U \cap V) \\ (\omega, \nu) &\longmapsto \omega|_{U\cap V} - \nu|_{U\cap V}. \end{aligned}$$

The composition of these two maps is visibly 0, and in fact the following is true:

Lemma 23.2.1. *The following is a short exact sequence of cochain complexes*

$$0 \longrightarrow \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \longrightarrow \Omega^*(U \cap V) \longrightarrow 0.$$

Proof. Exactness at $\Omega^p(M)$ amounts to the observation that a form on M is uniquely determined by its restrictions to U and V. Exactness at $\Omega^p(U) \oplus \Omega^p(V)$ amounts to the observation that a pair of forms ω on U and ν on V can be glued to a form on M if and only if $\omega|_{U\cap V} = \nu|_{U\cap V}$.

It is exactness at $\Omega^p(U \cap V)$ that is the hardest; we must show that every form on $U \cap V$ is a difference of forms on U and V. The problem is that a naive extension by 0 of $\omega \in \Omega^p(U \cap V)$ to U or V will not be smooth. To get around this, we will "cut off" ω appropriately before extending by 0. Let $\rho_U, \rho_V : M \to [0, 1]$ be a partition of unity subordinate to the open cover U, V. Then $\rho_V\omega$ can be extended by 0 to give a smooth p-form $\overline{\rho_V\omega}$ on U, and similarly $\rho_U\omega$ can be extended by 0 to give a smooth p-form $\overline{\rho_U\omega}$ on V. Then we can write ω as $\overline{\rho_V\omega} - (-\overline{\rho_U\omega})$, which exhibits ω as being in the image of the map $\Omega^p(U) \oplus \Omega^p(V) \to \Omega^p(U \cap V)$. □

Corollary 23.2.2 (Mayer–Vietoris)**.** *There is a long exact sequence*

$$\begin{tikzcd}
 & H^{p+1}(M) \arrow[r] & H^{p+1}(U) \oplus H^{p+1}(V) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^p(U) \oplus H^p(V) \arrow[r] & H^p(U \cap V) \arrow[ull, rounded corners]
\end{tikzcd}$$

In the Mayer–Vietoris long exact sequence, the left horizontal maps

$$H^p(M) \longrightarrow H^p(U) \oplus H^p(V)$$

are given by pullback along the inclusion $U \hookrightarrow M$ and $V \hookrightarrow M$. Similarly, the right horizontal maps

$$H^p(U) \oplus H^p(V) \longrightarrow H^p(U \cap V)$$

are the difference of the pullback along the inclusion $U \cap V \hookrightarrow U$ and the pullback along the inclusion $U \cap V \hookrightarrow V$. Finally, the boundary maps can be described rather explicitly; given $[\omega] \in H^p(U \cap V)$, one observes that $d(\overline{\rho_V\omega})$ and $d(-\overline{\rho_U\omega})$ coincide on $U \cap V$ and hence glue to a well-defined $(p+1)$-form on M. It will in fact be supported in $U \cap V$.

23.3 Applications

As an application of Mayer–Vietoris, we will now compute the cohomology of three basic examples of smooth manifolds. The guidelines for its use are as follows: you need to know the cohomology of three out of the following four manifolds: M, U, V, and $U \cap V$. Since we don't know many examples yet, these often tend to be contractible or are provided by an inductive hypothesis.

23.3.1 The cohomology groups of spheres

We start with spheres S^n.

Theorem 23.3.1. *The cohomology of S^n, $n \geqslant 1$, is given by*

$$H^*(S^n) = \begin{cases} \mathbb{R} & \text{if } * = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof will be induction over n, the initial case $n = 1$ having been completed two lectures ago. We can cover $S^n = \{(x_0, \ldots, x_n) \mid \sum x_i^2 = 1\}$ by two slightly enlarged hemispheres:

$$\begin{aligned} U &:= S^n \cap \{x \in \mathbb{R}^{n+1} \mid x_n > -\epsilon\}, \\ V &:= S^n \cap \{x \in \mathbb{R}^{n+1} \mid x_n < \epsilon\}. \end{aligned}$$

Then $U \cong \mathbb{R}^n$, $V \cong \mathbb{R}^n$ and $U \cap V \cong S^{n-1} \times \mathbb{R}$. Thus we get that both U and V have non-zero cohomology groups only in degree 0, while homotopy invariance says $H^*(U \cap V) \cong H^*(S^{n-1})$ which we know by the inductive hypothesis.

There are several cases for Mayer–Vietoris when we want to compute $H^p(S^n)$. Let us start with assume that $p > 1$. In this case we have

$$\begin{tikzcd}
 & H^p(S^n) \arrow[r] & H^p(U) \oplus H^p(V) = 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{p-1}(U) \oplus H^{p-1}(V) = 0 \arrow[r] & H^{p-1}(S^{n-1}) \arrow[ull, rounded corners] &
\end{tikzcd}$$

because $p-1, p \neq 0$. By exactness, we conclude that the pictured boundary map is an isomorphism, and thus

$$H^{p-1}(S^{n-1}) \longrightarrow H^p(S^n)$$

is an isomorphism as long as $p > 1$.

To deal with $p = 0, 1$, we inspect the relevant part of the long exact sequence:

$$\begin{tikzcd}
H^1(S^n) \arrow[r] & H^1(U) \oplus H^1(V) = 0 \arrow[r] & \cdots \\
H^0(S^n) \arrow[r] & H^0(U) \oplus H^0(V) \cong \mathbb{R}^2 \arrow[r, "(*)"] & H^0(S^{n-1}) = \mathbb{R} \arrow[ull, rounded corners]
\end{tikzcd}$$

Recalling the construction of the Mayer–Vietoris sequence the map $(*)$ is given by the difference of the restrictions, so by $\mathbb{R}^2 \ni (x, y) \mapsto x - y \in \mathbb{R}$. This is surjective with kernel $\mathbb{R}$. From this we see that $H^0(S^n) = \mathbb{R}$ and $H^1(S^n) = 0$. $\square$

23.3.2 The cohomology groups of punctured Euclidean spaces

We have computed $H^*(\mathbb{R}^n)$ already—it mostly vanishes—and $H^*(\mathbb{R}^n\backslash\{0\})$ follows from the previous computation since $\mathbb{R}^n\backslash\{0\} \cong S^{n-1} \times \mathbb{R}$—it has the same cohomology as S^{n-1}. What happens if you remove more points? It is easy for $n = 1$, as then removing points just disconnects $\mathbb{R}$ into some disjoint union of copies of $\mathbb{R}$.

Theorem 23.3.2. *Let X be a finite subset of $\mathbb{R}^n$, $n \geqslant 2$, then*

$$H^*(\mathbb{R}^n\backslash X) \cong \begin{cases} \mathbb{R} & \text{if } * = 0, \\ \mathbb{R}^{|X|} & \text{if } * = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is by induction over the cardinality r of X. The initial case $r = 1$ has been done above. For the induction step, we fix some $x \in X$ and cover $\mathbb{R}^n$ by $U = \mathbb{R}^n\backslash\{x\}$ and $V = \mathbb{R}^n\backslash(X\backslash\{x\})$. Their intersection $U \cap V$ is $\mathbb{R}^n\backslash X$.

We will not give the full Mayer–Vietoris sequence, but skip to the interesting part around degree $p = n-1$:

$$\begin{tikzcd} H^n(\mathbb{R}^n) = 0 \arrow[r] & \cdots \\ H^{n-1}(\mathbb{R}^n) = 0 \arrow[r] & H^{n-1}(U) \oplus H^{n-1}(V) = \mathbb{R} \oplus \mathbb{R}^{|X|-1} \arrow[r] & H^{n-1}(U \cap V) \cong \mathbb{R} \arrow[llu, rounded corners] \\ & & \cdots \arrow[llu, rounded corners] \end{tikzcd}$$

where we applied the inductive hypothesis to U and V respectively. We conclude that $H^{n-1}(\mathbb{R}\backslash X) \cong \mathbb{R}^{|X|}$. $\square$

23.3.3 The cohomology groups of $\mathbb{C}P^n$

Recall the complex projective plane $\mathbb{C}P^n$ is given by the quotient of the scaling action on non-zero vectors in $\mathbb{C}^n$:

$$(\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^\times.$$

That is, an element $[z_0 : \cdots : z_n] \in \mathbb{C}P^n$ is described by an $(n+1)$-tuple $(z_0, \ldots, z_n)$ of complex numbers, not all zero, up to scaling. Since $\mathbb{C}P^1$ is diffeomorphic to S^2, we already know its cohomology from Theorem 23.3.1. What happens for $\mathbb{C}P^n$, $n \geqslant 2$?

Theorem 23.3.3. *The cohomology of $\mathbb{C}P^n$, $n \geqslant 1$, is given by*

$$H^*(\mathbb{C}P^n) = \begin{cases} \mathbb{R} & \text{if } 0 \leqslant * \leqslant 2n \text{ is even,} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. The proof is by induction over n, the initial case $n = 1$ having been done before.

Let $U \subset \mathbb{C}P^n$ be the open subset consisting of those $[z_0 : \ldots : z_n]$ satisfying $|z_0|^2 + \ldots + |z_{n-1}|^2 > |z_n|^2$. By scaling the last coordinate by $1 - t$ with $t \in [0, 1]$, this deformation retracts onto $\mathbb{C}P^{n-1}$. Let $V \subset \mathbb{C}P^n$ be the open subset consisting of those $[z_0 : \cdots : z_n]$ with $z_n \neq 0$. By scaling the first n coordinates by $(1 - t)$ with $t \in [0, 1]$, this is seen to be contractible. Then $U \cap V$ is the open subset of those $[z_0 : \ldots : z_n]$ with $z_n \neq 0$ and $|z_0|^2 + \ldots + |z_n|^2 > |z_{n+1}|^2$. Such elements are uniquely represented by elements of the form $[w_0 : \ldots : w_{n-1} : 1]$ with $|w_0|^2 + \ldots + |w_{n-1}|^2 > 1$. This deformation retracts onto the subspace with $|w_0|^2 + \ldots + |w_{n-1}|^2 = 2$, which gives a sphere S^{2n-1}.

We will not give the full Mayer–Vietoris sequence, but skip to the interesting part around degree $p = 2n - 1$:

$$\cdots \longrightarrow H^{2n-1}(U) \oplus H^{2n-1}(V) \cong 0 \longrightarrow H^{2n-1}(U \cap V) \cong \mathbb{R} \longrightarrow H^{2n}(\mathbb{C}P^n) \longrightarrow H^{2n}(U) \oplus H^{2n}(V) = 0 \longrightarrow \cdots$$

In particular we get that $H^*(\mathbb{C}P^n) = H^*(\mathbb{C}P^{n-1})$ for $* < 2n$ and $H^{2n}(\mathbb{C}P^n) = \mathbb{R}$. $\square$

Multiplicative structures

Above we computed $H^*(S^n)$, $H^*(\mathbb{R}^n \backslash X)$, and $H^*(\mathbb{C}P^n)$ as graded $\mathbb{R}$-vector spaces. However, we actually know that these cohomology groups are a graded-commutative algebra. In the former two cases, this algebra structure is uniquely determined by the fact that it is compatible with the grading and that H^0 is generated by a unit; in both cases all products not involving a multiple of the unit vanish:

$$H^*(S^n) = \mathbb{R}[x_n]/(x_n^2),$$

the free polynomial ring on a generator x_n of degree n, modulo the ideal generated by x_n^2. Similarly,

$$H^*(\mathbb{R}^n \backslash X) = \mathbb{R}\left[y_{n-1}^{(x)} \,\middle|\, x \in X\right] / (y_{n-1}^{(x)} y_{n-1}^{(x')} \mid x, x' \in X),$$

with a collection of generators $y_{n-1}^{(x)}$ of degree $n - 1$, one for each element of X.

However, the algebra structure on $H^*(\mathbb{C}P^n)$ can not be determined this way. Once we establish Poincaré duality, we can prove that as an algebra

$$H^*(\mathbb{C}P^n) = \mathbb{R}[x_2]/(x_2^{n+1}),$$

with x_2 a generator in $H^2(\mathbb{C}P^n)$.

23.3.4 More examples

If you want to practice your proficiency with the Mayer–Vietoris sequence you can prove—at least additively— the following results (the convention is that a subscript on a generator denotes its degree).

Example 23.3.4. Recall the quaternionic projective plane $\mathbb{H}P^n$. Its cohomology is given by

$$H^*(\mathbb{H}P^n) = \mathbb{R}[y_4]/(y_4^{n+1}).$$

Here are some computations that require more advanced techniques than we have discussed so far:

Example 23.3.5. Let $U(2)$ be the Lie group of (2×2)-matrices with complex entries which are unitary, i.e. $A^\dagger = A$. Its cohomology is given by

$$H^*(U(2)) = \mathbb{R}[c_1, c_3]/(c_1^2, c_3^2).$$

Example 23.3.6. Recall the K3-manifold. Its cohomology is given by

$$H^*(K3) = \begin{cases} \mathbb{R} & \text{if } * = 0, \\ 0 & \text{if } * = 1, \\ \mathbb{R}^{22} & \text{if } * = 2, \\ 0 & \text{if } * = 3, \\ \mathbb{R} & \text{if } * = 4, \\ 0 & \text{otherwise.} \end{cases}$$

The multiplicative structure is determined by the bilinear map $H^2(K3) \times H^2(K3) \to H^4(K3) \cong \mathbb{R}$. In a suitable basis, it is given by the symmetric matrix

$$\begin{bmatrix} -\mathrm{id}_{19} & 0 \\ 0 & \mathrm{id}_3 \end{bmatrix}.$$

Remark 23.3.7. In fact, the Sullivan–Barge theorem tells you that the only restrictions on realizing a given finitely-generated graded-commutative $\mathbb{R}$-algebra H^* with $H^1 = 0$ as the cohomology of a manifold are (i) it satisfies Poincáre duality, and (ii) if the dimension is $4n$ it admits Pontryagin classes satisfying the congruences of the Hirzebruch signature theorem [FOT08, Theorem 3.2].

23.4 Problems

Problem 54 (Long exact sequence of a pair)**.** Suppose that $M \subset N$ is a smooth submanifold.

(a) Show that the differential of $\Omega^*(N)$ restricts to one on $\ker[\Omega^*(N) \to \Omega^*(M)]$,

We define the *relative cohomology* $H^*(N, M)$ as that of the cochain complex $\ker[\Omega^*(N) \to \Omega^*(M)]$.

(b) Prove that there is a long exact sequence

$$\begin{tikzcd}[column sep=large] & H^{p+1}(N,M) \arrow[r] & H^{p+1}(N) \oplus H^{p+1}(M) \arrow[r] & \cdots \\ \cdots \arrow[r] & H^p(N,M) \oplus H^p(N) \arrow[r] & H^p(M) \arrow[ul, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=2ex]\tikztostart.north) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \end{tikzcd}$$

This is known as the *long exact sequence of a pair*.

Problem 55 (Relative and compact-supported cohomology)**.** Suppose that M is a compact manifold with boundary ∂M. Prove there is an isomorphism

$$H^*(M, \partial M) \cong H^*_c(M \backslash \partial(M)).$$

Problem 56 (The compactly-supported cohomology of the Moebius strip)**.** Let M be the open Moebius strip. Use the Poincaré lemma and Mayer–Vietoris for compactly-supported cohomology to compute $H^*_c(M)$.

Problem 57 (Cohomology of compact oriented surfaces)**.** Recall that Σ_g denote a genus g surface. Use Mayer–Vietoris to compute $H^*(\Sigma_g)$.

Chapter 24

Qualitative applications of Mayer–Vietoris

So far we have only used Mayer–Vietoris to compute the cohomology of specific manifolds. Today we will use it to prove finite-dimensionality of de Rham cohomology and Poincaré duality. This is proven in of [BT82, Section 5].

24.1 De Rham cohomology is finite-dimensional

Suppose that M is a compact manifold, then we can find a finite cover by contractible subsets: take some collection of charts $\phi_\alpha \colon \mathbb{R}^k \supset U_\alpha \to V_\alpha \subset M$ which cover M, write each U_α as a union of open balls, and apply compactness. Using a trick from Riemannian geometry you can in fact do better and find a *good cover* in the following sense:

Definition 24.1.1. A finite open cover $U_1, \cdots, U_r$ of a topological space is *good* if for each non-empty subset $I \subset \{1, \ldots, r\}$, the open subset $U_I := \bigcap_{i \in I} U_i$ is either empty or diffeomorphic to $\mathbb{R}^n$.

Definition 24.1.2. A smooth manifold M is said to be *of finite type* if it admits a good open cover.

In particular, you can take $I = \{i\}$ to see that each U_i is contractible.

Example 24.1.3. A circle is of finite type; it has a good open cover by three intervals. More generally, a k-sphere is a finite type; it has a good open cover by $k+2$ open subsets, by taking neighborhoods of the k-simplices in the boundary $\partial\Delta^{k+1}$ of a standard $(k+1)$-simplex Δ^k (the convex hull of the basis vectors $e_0, \ldots, e_{k+1}$ in $\mathbb{R}^{k+2}$) For example, Δ^3 is the tetrahedron and slightly expanding the four faces of a tetrahedron gives a good open cover of S^2.

Remark 24.1.4. Definition 24.1.1 is slightly non-standard, chosen to simplify the proof of Poincaré duality. It is more common to define a good open cover to have U_I which are either empty or contractible. The minimal numbers of elements in a such good open cover is called the *covering type* [KW16]. Karoubi and Weibel used Mayer–Vietoris to prove that the k-sphere has no good open cover by $< k+2$ open subsets. You can prove this yourself, see Problem 58. Covering type has been largely unstudied and many open questions surrounding it; apparently the covering type of the Klein bottle is not known!

The following is proven in in [BT82, Theorem 5.1]:

Proposition 24.1.5. *Every compact manifold M is of finite type, i.e. admits a good open cover. Moreover, every open cover has a refinement to a good open cover.*

Theorem 24.1.6. *If M is of finite type, $H^*(M)$ is finite-dimensional.*

Corollary 24.1.7. *If M is compact, $H^*(M)$ is finite-dimensional.*

Proof of Theorem 24.1.6. First observe that since $H^p(M) = 0$ for $p > k$, the dimension of M, it suffices to prove that each $H^p(M)$ is finite-dimensional.

We prove the result by induction over the number r of open subsets in a good open cover. In the initial case $r = 1$, $M = U_1$ and U_1 is contractible, so by the homotopy invariance of de Rham cohomology $H^0(M) = \mathbb{R}$ and all other cohomology groups vanish.

For the induction step, suppose that M has a good open cover with r open subsets $U_1, \ldots, U_r$. Then M can be covered by two open subsets $U := U_1$ and $V := \bigcup_{i=2}^r U_i$. Then U is contractible, V has a good open cover by $r-1$ open subsets (namely $U_2, \ldots, U_r$), and $U \cap V$ has a good open cover by $r-1$ open subsets (namely $U_1 \cap U_2, \ldots, U_1 \cap U_r$). Now consider the Mayer–Vietoris long exact sequence

$$\begin{tikzcd}
 & H^p(M) \arrow[r] & H^p(U) \oplus H^p(V) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{p-1}(U) \oplus H^{p-1}(V) \arrow[r] & H^{p-1}(U \cap V) \cong \mathbb{R} \arrow[llu, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=3ex]\tikztostart.north) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] &
\end{tikzcd}$$

We deduce from it that for each $p \geqslant 0$, $H^p(M)$ has a surjection onto a subspace of $H^p(U) \oplus H^p(V)$ with kernel a subspace of $H^{p-1}(U \cap V)$. Both $H^p(U) \oplus H^p(V)$ and $H^{p-1}(U \cap V)$ are finite-dimensional by the inductive hypothesis, and hence so are these subspaces. This in turn implies $H^p(M)$ is finite-dimensional. $\square$

Remark 24.1.8. In fact, you can bound the dimension of $H^*(M)$ in terms of r as $\dim H^*(M) \leqslant 2^r$.

Some non-compact manifolds are of finite type, e.g. those which are the interior of a compact manifold with boundary. However, $H^*(M)$ is *not* finite-dimensional for a general non-compact manifold M.

Remark 24.1.9. Here is an alternative method to constructing a counterexample; suppose we have open subsets $U_1 \subset U_2 \subset \cdots$ of M such that $\bigcup_i U_i = M$, then it is a fact that $H^*(M)$ always surjects onto $\lim_i H^*(U_i)$. In fact, when all $H^*(U_i)$ are finite-dimensional this is an isomorphism. This follows from the Milnor sequence and the observation that inverse systems of finite-dimensional vector spaces are Mittag-Leffler. It is easy to construct examples of U_i where all maps $H^*(U_i) \to H^*(U_{i-1})$ surjective and the dimension increases, in which case the limit will be infinite-dimensional.

24.2 Poincaré duality

The following is a whirlwind tour of Poincaré duality, both its proof and applications.

24.2.1 Statement and proof

Recall that a bilinear form $V \times W \to \mathbb{R}$ is *non-degenerate* if (i) $\langle v, w\rangle = 0$ for all $w \in W$ if and only if $v = 0$, and (ii) $\langle v, w\rangle = 0$ for all $v \in V$ if and only if $w = 0$. Note that (i) says $V \to W^*$ is injective, and (ii) that $W \to V^*$ is injective. By counting dimensions, one proves the following lemma:

Lemma 24.2.1. *Suppose that V is finite-dimensional, then the following are equivalent:*

1. *the bilinear form $V \times W \to \mathbb{R}$ is non-degenerate,*
2. *$V \to W^*$ is an isomorphism,*
3. *$W \to V^*$ is an isomorphism.*

Under these conditions W is also finite-dimensional.

Recall that $H^*_c(M)$ denotes the compactly-supported de Rham cohomology, defined using compactly-supported forms instead of arbitrary forms.

Theorem 24.2.2 (Poincaré duality)**.** *If M is oriented of dimension k and of finite type, then the bilinear map*

$$\begin{aligned} \langle -, - \rangle \colon H^p(M) \times H^{k-p}_c(M) &\longrightarrow \mathbb{R} \\ ([\omega], [\nu]) &\longmapsto \int_M \omega \wedge \nu \end{aligned}$$

is non-degenerate.

As the compactly-supported cohomology of a compact manifold coincides with the ordinary cohomology, we get the following, making good on a promise from a previous lecture:

Corollary 24.2.3. *If M is compact oriented of dimension k with empty boundary, then there is an isomorphism $H^p(M) \cong H^{k-p}(M)$. In particular, if M is connected, $H^k(M) \cong \mathbb{R}$.*

It is easy to deduce more consequences. Recalling that if M is simply-connected then $H^1(M) = 0$, we conclude that:

Corollary 24.2.4. *If M is a compact oriented manifold of dimension k and simply-connected, then $H^{k-1}(M) = 0$.*

Using the isomorphism $H^*(M, \partial M) \cong H^*_c(M \backslash \partial M)$, one may deduce from Theorem 24.2.2 also a variant for manifolds with boundary.

Corollary 24.2.5 (Poincaré–Lefschetz duality)**.** *If M is compact oriented of dimension k with boundary ∂M, then $H^p(M) \cong H^{k-p}(M, \partial M)$.*

24.2.2 The proof of Poincaré duality

Before we start the proof we give a fundamental example:

Example 24.2.6. We know that $H^*(\mathbb{R}^k)$ is non-zero except for $* = 0$, in which case it is $\mathbb{R}$ generated by the class $[1]$ represented by the constant function with value 1. Similarly, $H_c^*(\mathbb{R}^k)$ is non-zero except for $* = k$ by the Poincaré lemma for compactly-supported cohomology, in which case it is $\mathbb{R}$ generated by the class $[\lambda(x)dx_1 \wedge \cdots dx_k]$ represented by any compactly-supported k-form $\lambda(x) \cdot dx_1 \wedge \cdots \wedge dx_k$ with $\lambda\colon \mathbb{R}^k \to \mathbb{R}$ a compactly-supported smooth function satisfying $\int_{\mathbb{R}^k} \lambda(x)dx_1 \cdots dx_k = 1$. Then the computation $\langle [1], [\lambda(x)dx_1 \wedge \cdots dx_k]\rangle = \int_{\mathbb{R}^k} \lambda(x)dx_1 \cdots dx_k = 1$ exhibits the bilinear form as being non-degenerate.

Proof of Theorem 24.2.2. Since the cohomology groups of a manifold of finite type are finite-dimensional, it suffices to prove that the slightly-modified map (we have added a sign)

$$\begin{aligned} \rho_M\colon H^p(M) &\longrightarrow (H_c^{k-p}(M))^* \\ \omega &\longmapsto \left(\nu \mapsto \epsilon(p)\int_M \omega \wedge \nu\right), \end{aligned}$$

is an isomorphism. Here, $\epsilon(p) = 1$ if $p \equiv 0, 1 \pmod 4$ and $\epsilon(p) = -1$ if $p \equiv 2, 3 \pmod 4$. The proof will be by induction over the number of elements r in the finite good cover $U_1, \ldots, U_r$. The initial case $r = 1$ has been done in Example 24.2.6.

For the induction step, we write M as the union of the two open subsets $U := U_1$ and $V := U_2 \cup \cdots \cup U_r$. Each of U, V and $U \cap V$ is oriented with a good open cover with either 1 or $r - 1$ elements, and thus the inductive hypothesis applies to them.

There are Mayer-Vietoris long exact sequences in cohomology and compactly-supported cohomology, the latter being reversed in direction with the maps not induced by pullback but by extension by 0:

$$\cdots \longrightarrow H^p(M) \longrightarrow H^p(U) \oplus H^p(V) \longrightarrow H^p(U \cap V) \longrightarrow H^{p+1}(M) \longrightarrow \cdots$$

and

$$\cdots \longleftarrow H_c^p(M) \longleftarrow H_c^p(U) \oplus H_c^p(V) \longleftarrow H_c^p(U \cap V) \longleftarrow H_c^{p-1}(M) \longleftarrow \cdots$$

The latter may be dualized to a long exact sequence

$$\cdots \longrightarrow H_c^p(M)^* \longrightarrow H_c^p(U)^* \oplus H_c^p(V)^* \longrightarrow H_c^p(U \cap V)^* \longrightarrow H_c^{p-1}(M)^* \longrightarrow \cdots$$

We can now write down integration maps from the long exact sequence for cohomology to this dual of the one for compactly-supported cohomology, a representative part of which is given by

$$\begin{tikzcd} H^p(M) \arrow[r] \arrow[d, "\rho_M"] & H^p(U) \oplus H^p(V) \arrow[r] \arrow[d, "\rho_U \oplus \rho_V"] & H^p(U \cap V) \arrow[r, "\partial"] \arrow[d, "\rho_{U \cap V}"] & H^{p+1}(M) \arrow[d, "\rho_M"] \\ H_c^{k-p}(M)^* \arrow[r] & H_c^{k-p}(U)^* \oplus H_c^{k-p}(V)^* \arrow[r] & H_c^{k-p}(U \cap V)^* \arrow[r, "\partial"] & H_c^{k-p-1}(M)^* \end{tikzcd}$$

We claim this diagram commutes. This is easy to see in the left two squares. For example, for the leftmost one it amounts to verifying that for each pair (ν_U, ν_V) of compactly-supported $(k-p)$-forms and each p-form ω on M, we have that

$$\begin{aligned}
(\rho_U \oplus \rho_V)(\omega|_U, \omega|_V)\Big((\nu_U, \nu_V)\Big) &= \epsilon(p) \int_U \omega|_U \wedge \nu_U + \epsilon(p) \int_V \omega|_V \wedge \nu_V \\
&= \epsilon(p) \int_M \omega \wedge (\nu_U + \nu_V) \\
&= (\rho_M)(\omega)\Big((\nu_U, \nu_V)\Big),
\end{aligned}$$

where in the last two lines we use the convention to denote the extension-by-zero of ν_U and ν_V to M be the same symbols.

It is the right square that is harder, as it involves boundary maps. For a $(p+1)$-form ω on $U \cap V$, recall that $\partial\omega$ is given by picking a partition of unity $\eta_U, \eta_V \colon M \to [0,1]$ subordinate to U, V and taking the $(p+1)$-form $\partial\omega$ given by $d(\eta_U \omega) = -d(\eta_V \omega)$. Similarly, the boundary map on compactly-supported cohomology sends a $(k-p-1)$-form ν to $d(\eta_U \nu) = -d(\eta_V \nu)$. Then we compute

$$\begin{aligned}
\rho_M(\partial\omega)\Big(\nu\Big) &= \epsilon(p+1) \int_M \partial\omega \wedge \nu \\
&= \epsilon(p+1) \int_U d(\eta_U \omega|_U) \wedge \nu \\
&= \epsilon(p+1) \int_U d(\eta_U) \wedge \omega|_U \wedge \nu,
\end{aligned}$$

where the second step uses that d is a derivation and ω is closed. We can in turn write this as

$$\begin{aligned}
&= (-1)^p \epsilon(p+1) \int_U \omega|_U \wedge d(\eta_U) \wedge \nu \\
&= (-1)^p \epsilon(p+1) \int_M \omega \wedge (d(\eta_U) \wedge \nu \\
&= (-1)^p \epsilon(p+1) \int_M \omega \wedge d(\eta_U \nu) \\
&= (-1)^p \epsilon(p+1) \int_M \omega \wedge \partial\nu \\
&= (-1)^p \epsilon(p+1) \epsilon(p) \rho_M(\omega)\Big(\partial\nu\Big).
\end{aligned}$$

Now we observe that there are two cases: if p is even then $\epsilon(p+1) = \epsilon(p)$, and if p is odd then $\epsilon(p+1) = -\epsilon(p)$, so this is exactly $\rho_M(\omega)(\partial\nu)$.

Thus we have a commutative diagram of long exact sequences with two-thirds of the vertical maps isomorphisms

$$\begin{tikzcd}
H^p(M) \arrow[r] \arrow[d, "\rho_M"] & H^p(U) \oplus H^p(V) \arrow[r] \arrow[d, "\rho_U \oplus \rho_V", "\cong"'] & H^p(U \cap V) \arrow[r] \arrow[d, "\rho_{U\cap V}"] & H^{p+1}(M) \arrow[d, "\rho_M", "\cong"'] \\
H_c^{k-p}(M)^* \arrow[r] & H_c^{k-p}(U)^* \oplus H_c^{k-p}(V)^* \arrow[r] & H_c^{k-p}(U \cap V)^* \arrow[r] & H_c^{k-p-1}(M)^*
\end{tikzcd}$$

It follows from Lemma 24.2.7 that the ρ_M must also be an isomorphism. $\square$

The following is a standard result in homological algebra (there is a much more general version):

Lemma 24.2.7 (5-lemma)**.** *If in a commutative diagrams of vector spaces*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

all vertical maps except f_3 are known to be isomorphisms, then f_3 must also be an isomorphism.

Proof. We shall prove that f_3 is injective, leaving the proof that it is surjective to the reader. Suppose that $f_3(x) = 0$, then in particular its image in B_4 vanishes. Since f_4 is an isomorphism, the image of x in A_4 must also vanish. By exactness, this means that x is the image of some $y \in A_2$. We know that $f_2(y)$ is mapped to 0 in B_3, so by exactness $f_2(y)$ is the image of some $z \in B_1$. Since f_1 and f_2 are isomorphisms, this means that there is some $w \in A_1$ which maps to $y \in A_2$. The element x must vanish, being in the image of a composition of two maps in an exact sequence, which is the zero map by exactness. $\square$

24.2.3 Multiplicative structure of the cohomology of $\mathbb{C}P^n$

In the previous lecture we computed $H^*(\mathbb{C}P^n)$ additively; it is $\mathbb{R}$ in degrees $* = 2i$ for $0 \leqslant i \leqslant n$ and vanishes otherwise. We now explain how to obtain the algebra structure.

Proposition 24.2.8. *As a graded-commutative $\mathbb{R}$-algebra, $H^*(\mathbb{C}P^n) = \mathbb{R}[x_2]/(x_2^{n+1})$.*

Proof. We prove this by induction over n, the case $n = 1$ being obvious as $H^*(\mathbb{C}P^1) = \mathbb{R}[x_2]/(x_2^2)$ for degree reasons. (Alternatively, you can use that $\mathbb{C}P^1$ is diffeomorphic to S^2.)

During the Mayer–Vietoris computation of the additive structure of $H^*(\mathbb{C}P^n)$ we learned that the inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induces an isomorphism on de Rham cohomology in degrees $* < 2n$. Thus $H^{2i}(\mathbb{C}P^n)$ is generated by x_2^i for $i < n$, and it remains to prove that x_2^n is non-zero, as then it necessarily generates the 1-dimensional group $H^{2n}(\mathbb{C}P^n)$. But that x_2^n is non-zero follows from Poincaré duality: there must exist a class y in $H^2(\mathbb{C}P^n)$ such that $y \cdot x_2^{n-1} \in H^{2n-2}(\mathbb{C}P^n)$ is a non-zero element of $H^{2n}(\mathbb{C}P^n)$ otherwise x_2^{n-1} would be the pairing as being non-degenerate. But y must be a non-zero multiple of x_2 and hence $x_2 \cdot x_2^{n-1} \neq 0$. $\square$

The multiplicative structure of cohomology groups can be used to prove results which can not be proven if you just know the additive structure. For example, the additive structure of $H^*(\mathbb{C}P^n)$ does not rule out that there may exist smooth maps $S^{2n} \to \mathbb{C}P^n \to S^{2n}$ whose composition is the identity. However, the multiplicative structure does:

Corollary 24.2.9. *If $n \geqslant 2$, there is no smooth map $S^{2n} \to \mathbb{C}P^n$ of non-zero degree.*

Proof. Such a map would need to be non-zero on H^{2n}, but since the map $H^*(\mathbb{C}P^n) = \mathbb{R}[x_2]/(x_2^{n+1}) \to H^*(S^{2n}) = \mathbb{R}[y_{2n}]/(y_{2n}^2)$ is a homomorphism, the value on the generator x_2^n of $H^{2n}(\mathbb{C}P^n)$ is the nth power of the value on the generator of x_2 of $H^2(\mathbb{C}P^n)$. But this is necessarily 0. □

24.3 Problems

Problem 58 (Bounds on non-zero cohomology groups)**.** Suppose that M has a good open cover by r subsets. Prove that the largest p such that $\widetilde{H}^p(M) \neq 0$ must be $\leqslant r-2$. (Hint: induct over r).

Problem 59 (Strengthening the 5-lemma)**.** How much can you weaken the assumptions on f_1, f_2, f_4, f_5 in Lemma 24.2.7 such that the conclusion still holds?

Problem 60 (The Künneth theorem)**.** In this problem you will use the techniques of this chapter to prove the Künneth theorem. Let M, N be smooth manifolds.

(a) Prove that given two cochain complexes C^* and D^*,

$$(C^* \otimes D^*)^p = \bigoplus_{k+l=p} C^k \otimes D^l \qquad d(x \otimes y) = d(x) \otimes y + (-1)^{|x|} x \otimes d(y)$$

is again a cochain complex. (The sign is another instance of the Koszul sign rule.)

(b) Prove that the map

$$\begin{aligned} H^*(C^*) \otimes H^*(D^*) &\longrightarrow H^*(C^* \otimes D^*) \\ [x] \otimes [y] &\longmapsto [x \otimes y] \end{aligned}$$

is well-defined and an isomorphism.

(c) Let $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ be the projections. Show that

$$\begin{aligned} \Omega^*(M) \otimes \Omega^*(N) &\longrightarrow \Omega^*(M \times N) \\ \omega \otimes \nu &\longmapsto \pi_1^*(\omega) \wedge \pi_2^*(\nu) \end{aligned} \tag{24.1}$$

is a map of cochain complexes.

Now suppose that N is of finite type.

(d) Prove that by induction over the number of elements in a good open cover that map (24.1) induces an isomorphism

$$H^*(M) \otimes H^*(N) \xrightarrow{\cong} H^*(M \times N).$$

(e) Compute $H^*(\mathbb{T}^n)$.

Chapter 25

The Thom isomorphism

We continue our discussion of Mayer–Vietoris and Poincaré duality with an intermediate form: the Thom isomorphism for a vector bundle, where one takes compact support only in the fibre direction.

25.1 Vertically compactly-supported cohomology

Let $\pi\colon E \to M$ be a d-dimensional smooth vector bundle over a k-dimensional smooth manifold M. Then it makes sense to consider those differential forms ω on M so that $\mathrm{supp}(\omega) \cap \pi^{-1}(K)$ is compact for all $K \subset M$ compact. This is preserved by the exterior derivative, so the subspaces of vertically compactly-supported p-forms assemble to a cochain complex

$$\Omega^*_{vc}(E).$$

To study these, we need another operation: integration along the fibre. Suppose that $E = \mathbb{R}^k \times \mathbb{R}^d$ then any p-form $\omega \in H^*(\mathbb{R}^k \times \mathbb{R}^d)$ is a sum of terms of two types: (I) $f(x,t)dx_I \wedge dt_J$ for $|J| = d$ (so $dt_J = dt_1 \wedge \cdots \wedge dt_d$)), (II) $f(x,t)dx_I \wedge dt_J$ for $|J| < d$. If we assume that f has compact support in the $\mathbb{R}^d$-direction for each $x \in \mathbb{R}^k$, then we can define

$$\pi_*(f(x,t)dx_I \wedge dt_J) = \begin{cases} \left(\int_{\mathbb{R}^d} f(x,t)dt\right) dx_I & \text{if } |J| = d, \\ 0 & \text{else.} \end{cases}$$

This obviously generalises to the case where $\mathbb{R}^k$ is replaced by open subset $U \subseteq \mathbb{R}^k$, and then using local trivialisations to the case that E is the total space of an oriented vector bundle; the orientations are necessary to define integration. We will leave the details to you. The result is a linear map

$$\pi_*\colon \Omega^*_{vc}(E) \longrightarrow \Omega^{*-d}(M)$$

called *integration along the fibre.*

Lemma 25.1.1. *We have that $\pi_* d = d\pi_*$.*

Proof. Since two forms on M are equal if they are equal locally, we may pick a local trivialisation and assume $E = \mathbb{R}^k \times \mathbb{R}^d$. The proof is now essentially that of the Poincaré lemma. We first verify this on forms of type (I):

$$\begin{aligned}\pi_* d\left(f(x,t)dx_I \wedge dt_J\right) &= \pi_*\left(\sum_{i=1}^k \frac{\partial f(x,t)}{\partial x_i} dx_i \wedge dx_I \wedge dt_J\right)\\ &= \left(\int_{\mathbb{R}^d} \frac{\partial f(x,t)}{\partial x_i} dt\right) dx_i \wedge dx_I\\ &= \frac{\partial\left(\int_{\mathbb{R}^d} f(x,t)dt\right)}{\partial x_i} dx_i \wedge dx_I\\ &= d\pi_*\left(f(x,t)dx_I \wedge dt_J\right).\end{aligned}$$

We next observe that it both sides clearly send a form of type (II) to zero unless $|J| = d-1$, i.e. $dt_J = dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_d$. In this case we use in the first equality that only the term that takes a partial derivative with respect to t_i survives π_*

$$\pi_* d\left(f(x,t)dx_I \wedge dt_J\right) = \pi_*\left(\frac{\partial f(x,t)}{\partial t_i} dt_i \wedge dx_I \wedge dt_J\right) = 0$$

where the last equality is obtained by using Fubini's theorem to first integrate the t_i-coordinate, and that the result is zero by the fundamental theorem of algebra combined with the $f(x,t)$ having compact support in the t_i-direction when fixing the remaining coordinates. $\square$

Thus there is an induced map on vertically compactly-supported cohomology:

$$\pi_* \colon H^*_{vc}(E) \longrightarrow H^{*-d}(M).$$

It has the following property

Proposition 25.1.2 (Projection formula)**.** *Suppose that $\pi \colon E \to M$ is an oriented vector bundle of dimension d. Then for $\omega \in \Omega^p(M)$ and $\nu \in \Omega^q(E)$ we have*

$$\pi_*(\pi^*\omega \wedge \nu) = \omega \wedge \pi_*\nu.$$

Proof. Since two forms on M are equal if they are equal locally, we may pick a local trivialisation and assume $E = \mathbb{R}^k \times \mathbb{R}^d$. If ν is of type (II) then the left side vanishes by definition, and so does the right side since $\pi^*\omega \wedge \nu$ is also of type (II). If $\nu = f(x,t)dx_I \wedge dt_J$ is of type (I) and $\omega = g(x)dx_{I'}$ then $\pi^*\omega \wedge \nu$ is $g(x)f(x,t)dx_{I'} \wedge dx_I \wedge dt_J$ and we see that

$$\begin{aligned}\pi_*(\pi^*\omega \wedge \nu) &= \left(\int_{\mathbb{R}^d} g(x)f(x,t)dt_J\right) dx_{I'} \wedge dx_I\\ &= g(x)x_{I'} \wedge \left(\int_{\mathbb{R}^d} f(x,t)dt_J)dx_I\right)\\ &= \omega \wedge \pi_*\nu.\end{aligned}$$

$\square$

25.2 The Thom isomorphism

The proof of the Poincaré lemma in compactly-supported cohomology gives that:

Theorem 25.2.1. *The map*

$$\pi_*\colon H^*_{vc}(\mathbb{R}^k \times \mathbb{R}^d) \longrightarrow H^{*-d}(\mathbb{R}^k)$$

is an isomorphism.

For $U \subset M$ write $E_U = \pi^{-1}(U)$. It is easy to see that the sequence of chain complexes

$$0 \longrightarrow \Omega^*_{vc}(E_M) \longrightarrow \Omega^*_{vc}(E_U) \oplus \Omega^*_{vc}(E_V) \longrightarrow \Omega^*_{vc}(E_{U\cap V}) \longrightarrow 0$$

is short exact, so we get a Mayer–Vietoris long exact sequence for vertically compactly-supported cohomology. These are the necessary ingredients for:

Theorem 25.2.2 (Thom)**.** *If $E \to M$ is an oriented vector bundle of dimension d and M is of finite type, then the map*

$$\pi_*\colon H^*_{vc}(E) \longrightarrow H^{*-d}(M)$$

is an isomorphism.

Proof. The proof is once more by induction over the number of elements in a good open cover of M; we may assume that the vector bundle trivialises over the elements in the good open cover. This can be proven either by showing that any open cover contains a good open cover, or by proving that vector bundles over $\mathbb{R}^k$ are always trivialisable. The initial case is covered by the Poincaré lemma above so it remains to do the induction step. As usual we set $U = U_0$ and $V = U_1 \cup \cdots \cup U_r$ and it suffices by the five-lemma to prove that there is a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^*_{vc}(E_M) \arrow[r] \arrow[d, "\pi^M_*"] & H^*_{vc}(E_U) \oplus H^*_{vc}(E_V) \arrow[r] \arrow[d, "\pi^U_* \oplus \pi^V_*"] & H^*_{vc}(E_{U\cap V}) \arrow[r] \arrow[d, "\pi^{U\cap V}_*"] & H^{*+1}_{vc}(E_M) \arrow[r] \arrow[d, "\pi^M_*"] & \cdots \\
\cdots \arrow[r] & H^{*-d}(M) \arrow[r] & H^{*-d}(U) \oplus H^{*-d}(V) \arrow[r] & H^{*-d}(U \cap V) \arrow[r] & H^{*-d+1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

and it is easy to see that the left and middle square commute, but for the right square we need to verify that fibre integration commutes with the connecting homomorphisms:

$$\pi^M_* \partial\omega = \pi^{U\cap V}_*(\pi^* d\eta_U \wedge \omega|_{U\cap V}) = d\eta_U \wedge \pi^{U\cap V}_* \omega|_{U\cap V} = \partial \pi^{U\cap V}_*(\omega)$$

where the middle equation uses the projection formula. □

In particular, corresponding to $1 \in H^0(M)$ there is a vertically compactly-supported cohomology class $[\mathrm{Th}(\pi)] \in H^d_{\mathrm{vc}}(E)$. A vertically compactly-supported representative d-form $\mathrm{Th}(\pi)$ has the property that its integral over each fibre equals 1, and this property determines it uniquely up to exterior derivatives of vertically compactly-supported $(d-1)$-forms. Note that there is a map

$$\begin{aligned} H^{*-d}(M) &\longrightarrow H^*_{\mathrm{vc}}(E) \\ [\omega] &\longmapsto [\pi^*\omega \wedge \mathrm{Th}(\pi)] \end{aligned}$$

Corollary 25.2.3. *This map is inverse to* π_*.

Proof. We simply compute

$$\pi_*(\pi^*\omega \wedge \mathrm{Th}(\pi)) = \omega \wedge \pi_*\mathrm{Th}(\pi) = \omega,$$

using that $\pi_*\mathrm{Th}(\pi) = 1$. □

Chapter 26

Čech cohomology

We will now generalise the Mayer–Vietoris principle to a combinatorial method to compute de Rham cohomology: the Čech complex. This is also explained in [BT82, Section 8].

26.1 Double cochain complexes

Recall a cochain complex is given by a collection of $\mathbb{N}$-indexed $\mathbb{R}$-vector spaces C^* with differentials $d\colon C^* \to C^{*+1}$ satisfying $d^2 = 0$:

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots .$$

A double complex is $\mathbb{N}^2$-indexed and consequently has differentials going in two independent directions:

Definition 26.1.1. A *double cochain complex* is a collection of $\mathbb{N}^2$-indexed $\mathbb{R}$-vector spaces $C^{*,*}$ with horizontal differential $d\colon C^{*,*} \to C^{*+1,*}$ and vertical differential $\delta\colon C^{*,*} \to C^{*,*+1}$ satisfying $d^2 = 0$, $\delta^2 = 0$, and $\delta d = d\delta$. It looks like:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \ddots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^{0,2} & \xrightarrow{d} & C^{1,2} & \xrightarrow{d} & C^{2,2} & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^{0,1} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & C^{2,1} & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^{0,0} & \xrightarrow{d} & C^{1,0} & \xrightarrow{d} & C^{2,0} & \xrightarrow{d} & \cdots
\end{array}$$

The hypothesis that d and δ commute allows us to extract three functor cochain complex from a double cochain complex. The first is given by by summing along diagonals:

Definition 26.1.2. The *total cochain complex* $\mathrm{Tot}^*(C^{*,*})$ of a double cochain complex $C^{*,*}$ has entries given by

$$\mathrm{Tot}^p(C^{*,*}) := \bigoplus_{i+j=p} C^{i,j} \qquad \text{for } p \in \mathbb{N},$$

and differential $D\colon \mathrm{Tot}^p(C^{*,*}) \to \mathrm{Tot}^{p+1}(C^{*,*})$ given on the term $C^{i,j}$ by $d + (-1)^i\delta$.

The sign in the definition of D is necessary: on $C^{i,j}$ we have

$$D^2 = d^2 + (-1)^i d\delta + (-1)^{i+1}\delta d - \delta^2 = 0,$$

and without the sign the middle terms would not have cancelled and the result would be $d\delta + \delta d = 2d\delta$ instead.

Example 26.1.3. An element in $\mathrm{Tot}^*(C^{*,*})$ is given by a collection $a = (a_{i,j})_{i+j=p}$ of elements $a_{i,j} \in C^{i,j}$ and this is in the kernel of D if and only if

- $\delta a_{0,p} = 0$,
- $da_{i-1,j} = (-1)^i\delta a_{i,j-1}$ for all $i + j = p + 1$ with $i, j > 0$,
- $da_{p,0} = 0$.

The second is the *vertical edge*. As for a double cochain complex d and δ commute, the latter restricts for each row $C^{*,p}$ to a map

$$\delta\colon C^{-1,p} := \ker(d\colon C^{0,p} \to C^{1,p}) \longrightarrow C^{-1,p+1} := \ker(d\colon C^{0,p+1} \to C^{1,p+1})$$

which still satisfies $\delta^2 = 0$: the result is another cochain complex $C^{-1,*}$.

The third is the *horizontal edge*. As for the vertical edge, we can use the columns to extract a cochain complex $C^{*,-1}$ with entries $C^{p,-1} := \ker(\delta\colon C^{p,0} \to C^{p,1})$ and differential the restriction of d.

By construction, the inclusions induce a map of cochain complexes

$$C^{-1,*} \longrightarrow \mathrm{Tot}^*(C^{*,*}).$$

The following is once more a diagram chase and best followed on paper.

Theorem 26.1.4. *Suppose that the extended cochain complex of the columns*

$$\cdots \longrightarrow 0 \longrightarrow C^{-1,p} \xrightarrow{\text{inc}} C^{0,p} \xrightarrow{\delta} C^{1,p} \xrightarrow{\delta} \cdots$$

is exact for all $p \geqslant 0$. Then the inclusion $C^{-1,*} \to \mathrm{Tot}^*(C^{*,*})$ induces an isomorphism on cohomology.

Proof. We first prove that it is surjective. Suppose that $a = (a_{i,j})_{i+j=p} \in \mathrm{Tot}^p(C^{*,*})$ is in the kernel of D. We claim up to the image of D, we can replace a by a' satisfying $a'_{i,j} = 0$ for $i > 0$: then $da'_{p,0} = 0$ and $\delta a'_{p,0} = 0$, so it is the image of a cohomology class in $H^p(C^{-1,*})$ we are done. This is done inductively: suppose that $a_{i,j} = 0$ for $i > r$ with $r > 0$ then we have that $da_{r,p-r} = 0$, so by the hypothesis there exists an $b_{r-1,p-r} \in C^{r-1,p-r}$ so that $db_{r-1,p-r} = a_{r,p-r}$. Considering $b_{r-1,p-r}$ as an element of $\mathrm{Tot}^{p-1}(C^{*,*})$ we consider

$$a' = a - Db_{r-1,p-r}.$$

It has the analogous property with r replaced $r-1$ as we have killed the $(r, p-r)$-term at the cost of replacing the $(r-1, p+r-1)$-term with $a_{r-1,p+r-1} - (-1)^{r-1}\delta b_{r-1,p+r-1}$.

We next prove that it is injective. Suppose that $x \in C^{-1,p}$ is in the kernel of δ and that there exists a $b \in \mathrm{Tot}^{p-1}(C^{*,*})$ so that $x = Db$. We claim that up to the image of D, we can replace b by b' satisfying $b'_{i,j} = 0$ for $j > 0$: then $\delta b'_{0,p-1} = x$ and $db'_{0,p-1} = 0$, so x represents the zero cohomology class in $H^p(C^{-1,*})$. This is similarly done inductively in a similar manner and we leave the proof to the reader. □

There is similarly an inclusion of cochain complexes

$$C^{*,-1} \longrightarrow \mathrm{Tot}^*(C^{*,*})$$

and if the extended cochain complex of the rows

$$\cdots \longrightarrow 0 \longrightarrow C^{p,-1} \xrightarrow{\text{inc}} C^{p,0} \xrightarrow{d} C^{,p1} \xrightarrow{d} \cdots$$

is exact for all $p \geqslant 0$, then the inclusion $C^{*,-1} \to \mathrm{Tot}^*(C^{*,*})$ induces an isomorphism on cohomology. Let us combine these two facts:

Corollary 26.1.5. *Suppose that the extended cochain complexes of the columns and rows*

$$\cdots \longrightarrow 0 \longrightarrow C^{-1,p} \xrightarrow{\text{inc}} C^{0,p} \xrightarrow{\delta} C^{1,p} \xrightarrow{\delta} \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow C^{p,-1} \xrightarrow{\text{inc}} C^{p,0} \xrightarrow{d} C^{,p1} \xrightarrow{d} \cdots$$

are exact for all $p \geqslant 0$. Then $H^*(C^{-1,*}) \cong H^*(C^{*,-1})$.

26.2 The Čech-to-de Rham complex

We will now apply these ideas to give a variant of the Mayer–Vietoris principle for an arbitrary open cover rather than an open cover by two open subsets. We say "principle" here because we stay shy of extracting the analogue of the Mayer–Vietoris long exact sequence, which would require a digression into spectral sequences.

26.2.1 The Čech-to-de Rham complex

Let $\mathcal{U}$ be an open cover a smooth manifold M. For each finite collection $\alpha = \{\alpha_0, \ldots, \alpha_p\}$ of indices we can form the intersection

$$U_{\alpha_0,\ldots,\alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \subseteq M.$$

Note that there is an inclusion $U_{\alpha_0,\ldots,\alpha_p} \hookrightarrow U_{\beta_0,\ldots,\beta_p}$ when $\{\beta_0,\ldots,\beta_p\} \subseteq \{\alpha_0,\ldots,\alpha_p\}$.

Definition 26.2.1. The *Čech-to-de Rham complex* $C^*(\mathcal{U}, \Omega^*)$ is the double cochain complex with entries given by

$$C^q(\mathcal{U}, \Omega^p) := \left\{ (\omega)_{\alpha_0,\ldots,\alpha_q} \;\middle|\; \begin{array}{c} (\omega)_{\alpha_0,\ldots,\alpha_q} = \\ (-1)^{\sigma}(\omega)_{\alpha_{\sigma(0)},\ldots,\alpha_{\sigma(q)}} \\ \text{for all } \sigma \in S_{q+1} \end{array} \right\} \subset \prod_{|\alpha|=q+1} \Omega^p(U_\alpha),$$

with horizontal differential d given by

$$(d\omega)_{\alpha_0,\dots,\alpha_q} = d(\omega_{\alpha_0,\dots,\alpha_q})$$

and the vertical differential δ given by

$$(\delta\omega)_{\alpha_0,\dots,\alpha_q} = \sum_{i=0}^{q} (-1)^i (\omega)_{\alpha_0,\dots,\widehat{\alpha}_i,\dots,\alpha_q}|_{U_{\alpha_0,\dots,\alpha_q}}.$$

Notation 26.2.2. From now on, we will often drop the restrictions $(-)|_{U_{\alpha_0,\dots,\alpha_q}}$ to simplify the notation.

Remark 26.2.3. There are two variations of the Čech-to-de Rham double cochain complex which give isomorphic cohomology groups: (i) you can remove the anti-symmetry condition, (ii) you can order the indexing set of the open cover $\mathfrak{U}$ and only take the terms $(\omega)_{\alpha_0<\dots<\alpha_q}$.

This is well-defined: d clearly preserves the anti-symmetry condition in the definition of $C^*(\mathfrak{U}, \Omega^*)$ and δ does by a straightforward computation.

Example 26.2.4. If $\mathfrak{U}$ has two elements U and V, then $C^*(\mathfrak{U}, \Omega^*)$ is given by

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
\Omega^0(U\cap V) \arrow[u, "\delta"] \arrow[r, "d"] & \Omega^1(U\cap V) \arrow[u, "\delta"] \arrow[r, "d"] & \Omega^2(U\cap V) \arrow[u, "\delta"] \arrow[r, "d"] & \cdots \\
\Omega^0(U)\times\Omega^0(V) \arrow[u, "\delta"] \arrow[r, "d"] & \Omega^1(U)\times\Omega^1(V) \arrow[u, "\delta"] \arrow[r, "d"] & \Omega^2(U)\times\Omega^2(V) \arrow[u, "\delta"] \arrow[r, "d"] & \cdots
\end{tikzcd}$$

with vertical maps given by $(\omega, \nu) \mapsto \omega|_{U\cap V} - \nu|_{U\cap V}$. Here we implicitly identify the subgroup of the form $(\omega, -\omega) \in \Omega^i(U\cap V) \times \Omega^i(V\cap U)$ with $\Omega^i(U\cap V)$.

Let us verify the claim made in the definition:

Lemma 26.2.5. *$C^*(\mathfrak{U}, \Omega^*)$ is a double cochain complex.*

Proof. It is clear that $d^2 = 0$ (since the exterior derivative is a differential) and that $d\delta = \delta d$ (since the exterior derivative commutes with pullback, here appearing in the guise of restriction). It remains to see that $\delta^2 = 0$, which follows form

$$\begin{aligned}
(\delta^2\omega)_{\alpha_0,\dots,\alpha_q} &= \sum_{i=0}^{q} (-1)^i (\delta\omega)_{\alpha_0,\dots,\widehat{\alpha}_i,\cdots,\alpha_q} \\
&= \sum_{0\leqslant i<j\leqslant q} (-1)^i(-1)^{j-1}(\delta\omega_{\alpha_0,\dots,\widehat{\alpha}_i,\dots,\widehat{\alpha}_j,\dots,\alpha_q}) \\
&\quad + \sum_{0\leqslant j<i\leqslant q} (-1)^i(-1)^j(\delta\omega_{\alpha_0,\dots,\widehat{\alpha}_j,\dots,\widehat{\alpha}_i,\dots,\alpha_q})
\end{aligned}$$

and noting that a given pair of omissions appears twice with opposite sign. □

26.2.2 A generalised Mayer–Vietoris principle

On the one hand, to give a p-form on M is the same as giving p-forms on each of the U_α that agree on the overlaps $U_{\alpha<\beta}$. Thus we have that restriction induces an isomorphism

$$\Omega^p(M) \xrightarrow{\cong} C^{-1}(\mathcal{U}, \Omega^p) := \ker(\delta\colon C^0(\mathcal{U}, \Omega^p) \to C^1(\mathcal{U}, \Omega^q)),$$

and the restriction of the horizontal differential d of $C^*(\mathcal{U}, \Omega^*)$ to these kernels corresponds under this isomorphism to the exterior derivative of p-forms on M. To make use of this, we need to the following:

Lemma 26.2.6. *The extended cochain complex of the columns*

$$\cdots \longrightarrow 0 \longrightarrow C^{-1}(\mathcal{U}, \Omega^p) \longrightarrow C^0(\mathcal{U}, \Omega^p) \longrightarrow C^2(\mathcal{U}, \Omega^p) \longrightarrow \cdots$$

is exact for all $p \geqslant 0$.

Proof. To prove that any $(\omega) \in C^q(\mathcal{U}, \Omega^p)$ that lies in kernel of δ also lies in the image of δ, we pick a partition of unity η_α subordinate to the open cover $\mathcal{U}$ and consider $\nu \in C^{q-1}(\mathcal{U}, \Omega^p)$ given by

$$(\nu)_{\alpha_0,\ldots,\alpha_{q-1}} := \sum_\alpha \eta_\alpha(\omega)_{\alpha,\alpha_0,\ldots,\alpha_{q-1}}$$

which is well-defined as a locally finite sum of p-forms. The hypothesis that $(\delta\omega) = 0$ implies that

$$(\delta\omega)_{\alpha,\alpha_0,\ldots,\alpha_q} = (\omega)_{\alpha_0,\ldots,\alpha_q} + \sum_{i=0}^{q}(-1)^{i+1}(\omega)_{\alpha,\alpha_0,\ldots,\widehat{\alpha}_i,\ldots,\alpha_q} = 0.$$

Using this we check that

$$\begin{aligned}
(\delta\nu)_{\alpha_0,\ldots,\alpha_q} &= \sum_{i=0}^{q}(\nu)_{\alpha_0,\ldots,\widehat{\alpha}_i,\ldots\alpha_q} \\
&= \sum_{i=0}^{q}(-1)^i\sum_\alpha \eta_\alpha(\omega)_{\alpha,\alpha_0,\ldots,\widehat{\alpha}_i,\ldots\alpha_q} \\
&= \sum_\alpha \eta_\alpha(\omega)_{\alpha_0,\ldots,\alpha_q} = (\omega)_{\alpha_0,\ldots,\alpha_q}.
\end{aligned}$$

This proves that (ω) is in the image of δ. □

Theorem 26.2.7 (Generalised Mayer–Vietoris principle)**.** *For any open cover $\mathcal{U}$ of a smooth manifold M, the restriction map*

$$\Omega^*(M) \longrightarrow C^*(\mathcal{U}, \Omega^*)$$

induces an isomorphism on cohomology.

To obtain the Mayer–Vietoris theorem from this we need to perform some further manipulations: filter the double cochain complex to extract a Mayer–Vietoris spectral sequence that for an open cover by two open subsets degenerates to the Mayer–Vietoris long exact sequence. We will not do this here, but it may serve as motivation to learn about spectral sequences, e.g. from [BT82, Chapter III].

26.2.3 The Čech complex

On the other hand, the smooth functions in $\Omega^0(K)$ that lie in the kernel of the exterior derivative are exactly those that are locally constant. Denote locally constant $\mathbb{R}$-valued functions on U by $\underline{\mathbb{R}}(U)$, so that assigning to each component the value of a locally function on it, we get an isomorphism

$$\underline{\mathbb{R}}(U) \xrightarrow{\cong} \mathbb{R}^{\pi_0 U}.$$

The inclusion of locally constant functions induces an isomorphism

$$\begin{array}{c}\prod_{|\alpha|=q+1} \underline{\mathbb{R}}(U_{\alpha_0,\ldots,\alpha_q}) \supset \left\{ (f)_{\alpha_0,\ldots,\alpha_q} \;\middle|\; \begin{array}{c} (f)_{\alpha_0,\ldots,\alpha_q} = \\ (-1)^\sigma (f)_{\alpha_{\sigma(0)},\ldots,\alpha_{\sigma(q)}} \\ \text{for all } \sigma \in S_{q+1} \end{array} \right\} \\ \big\downarrow \cong \\ C^q(\mathfrak{U}, \Omega^{-1}) := \ker(d\colon C^q(\mathfrak{U}, \Omega^0) \to C^q(\mathfrak{U}, \Omega^1))\end{array}$$

and the vertical differential δ restricts to it. The resulting cochain complex is quite combinatorial, as it only depends on the sets of path components of the intersections of elements of the open cover. We will give it a name:

Definition 26.2.8. The Čech complex $\check{C}^*_{\mathfrak{U}}(M; \underline{\mathbb{R}})$ of an open cover $\mathfrak{U}$ of a smooth manifold M has entries given by

$$\check{C}^p_{\mathfrak{U}}(M; \underline{\mathbb{R}}) := \left\{ (f)_{\alpha_0,\ldots,\alpha_q} \;\middle|\; \begin{array}{c} (f)_{\alpha_0,\ldots,\alpha_q} = \\ (-1)^\sigma (f)_{\alpha_{\sigma(0)},\ldots,\alpha_{\sigma(q)}} \\ \text{for all } \sigma \in S_{q+1} \end{array} \right\} \subseteq \prod_{|\alpha|=p+1} \underline{\mathbb{R}}(U_{\alpha_0,\ldots,\alpha_p})$$

with differential given by $(\delta f)_{\alpha_0,\ldots,\alpha_q} = \sum_{i=0}^q (-1)^i (f)_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_q}|_{U_{\alpha_0,\ldots,\alpha_q}}$. We will write

$$\check{H}^*_{\mathfrak{U}}(M; \underline{\mathbb{R}}) := H^*(\check{C}^*_{\mathfrak{U}}(M; \underline{\mathbb{R}})).$$

It is not true in general that the extended rows are exact, but this *is* the case if each $U_{\alpha_0 < \cdots < \alpha_p}$ is a disjoint union of contractible components, by our computation of the de Rham cohomology of contractible manifolds:

Lemma 26.2.9. *Suppose each $U_{\alpha_0,\ldots,\alpha_p}$ is a disjoint union of contractible components. Then the extended cochain complexes of the columns*

$$\cdots \longrightarrow 0 \longrightarrow C^q(\mathfrak{U}, \Omega^{-1}) \longrightarrow C^q(\mathfrak{U}, \Omega^0) \longrightarrow C^q(\mathfrak{U}, \Omega^1) \longrightarrow \cdots$$

are exact for all $q \geqslant 0$.

Corollary 26.2.10. *For an open cover $\mathfrak{U}$ of a smooth manifold M as in Lemma 26.2.9, the inclusion map*

$$\check{C}^*_{\mathfrak{U}}(M; \underline{\mathbb{R}}) \longrightarrow C^*(\mathfrak{U}, \Omega^*)$$

induces an isomorphism on cohomology.

Combining this with Theorem 26.2.7 we get:

Theorem 26.2.11. *For an open cover $\mathfrak{U}$ of a smooth manifold M as in Lemma 26.2.9, there is an isomorphism $\check{H}^*_{\mathfrak{U}}(M;\underline{\mathbb{R}}) \cong H^*_{dR}(M)$.*

By the existence of good open covers, for compact M there always exist open covers $\mathfrak{U}$ so that the intersections $U_{\alpha_0,\ldots,\alpha_p}$ are not merely disjoint union of contractible components but in fact either empty or diffeomorphic to $\mathbb{R}^d$. By the above theorem, the cohomology of M is determined only by the combinatorics of the inclusions between components of these intersections.

26.3 Čech cohomology of (pre)sheaves

We end with a discussion of a general setting for the construction of Čech cohomology. The construction of the Čech complex only requires that (i) we can assign to each open subset U of M a $\mathbb{R}$-vector space $F(U)$, (ii) for an inclusions $U \subseteq V$ we have a restriction map

$$\mathrm{res}^V_U \colon F(V) \longrightarrow F(U)$$

so that for an pair of inclusion $U \subseteq V \subseteq W$ we have

$$\mathrm{res}^W_V \circ \mathrm{res}^V_U = \mathrm{res}^W_U.$$

This is conveniently encoded in terms of category theory. Let $\mathrm{Open}(M)$ be the category whose objects are open subsets $U \subseteq M$ and a unique morphism $U \to V$ when $U \subseteq V$, then the objects F we just described are the same as:

Definition 26.3.1. A *presheaf on M* is a functor

$$F \colon \mathrm{Open}(M)^{\mathrm{op}} \longrightarrow \mathrm{Vect}_{\mathbb{R}}.$$

Example 26.3.2. There is a presheaf $\underline{\mathbb{R}} \colon \mathrm{Open}(M) \to \mathrm{Vect}_{\mathbb{R}}$ which assigns to U the $\mathbb{R}$-vector space of locally constant $\mathbb{R}$-valued functions.

Example 26.3.3. For each $p \geqslant 0$ there is a presheaf $\Omega^q \colon \mathrm{Open}(M) \to \mathrm{Vect}_{\mathbb{R}}$ which assigns to U the $\mathbb{R}$-vector space of q-forms on U.

This is all the data we need to define the Čech cochain complex:

Definition 26.3.4. Let $F \colon \mathrm{Open}(M) \to \mathrm{Vect}_{\mathbb{R}}$ be a presheaf and $\mathfrak{U}$ be an open cover. Then the Čech cochain complex

$$\check{C}^p_{\mathfrak{U}}(M;F) := \left\{ (s)_{\alpha_0,\ldots,\alpha_q} \;\middle|\; \begin{array}{c} (s)_{\alpha_0,\ldots,\alpha_q} = \\ (-1)^{\sigma}(s)_{\alpha_{\sigma(0)},\ldots,\alpha_{\sigma(q)}} \\ \text{for all } \sigma \in S_{q+1} \end{array} \right\} \subseteq \prod_{|\alpha|=p+1} F(U_{\alpha_0,\ldots,\alpha_p})$$

with differential given by $(\delta s)_{\alpha_0,\ldots,\alpha_q} = \sum_{i=0}^q (-1)^i (s)_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_q}|_{U_{\alpha_0,\ldots,\alpha_q}}$. We will write

$$\check{H}^*_{\mathfrak{U}}(M;F) := H^*(\check{C}^*_{\mathfrak{U}}(M;F)).$$

Remark 26.3.5. There are two variations of the Čech cochain complex which give isomorphic cohomology groups: (i) you can remove the anti-symmetry condition, (ii) you can order the indexing set of the open cover $\mathcal{U}$ and only take the terms $(s)_{\alpha_0 < \cdots < \alpha_q}$.

Both the aforementioned examples of presheaves have the property that given elements $f_\alpha \in F(U_\alpha)$ whose restrictions to $F(U_\alpha \cap U_{\alpha'})$ agree, we can uniquely glue these to an $f \in F(U)$ where $U = \cup_\alpha U_\alpha$. A presheaf with such a gluing property is called a *sheaf*:

Definition 26.3.6. A presheaf $F\colon \mathrm{Open}(M)^{\mathrm{op}} \to \mathrm{Vect}_{\mathbb{R}}$ is a *sheaf* if for each open cover $\mathcal{U}$ of U the following is exact

$$0 \longrightarrow F(U) \longrightarrow \prod_{\alpha \in A} F(U_\alpha) \longrightarrow \prod_{(\alpha,\alpha') \in A^2} F(U_\alpha \cap U_{\alpha'}).$$

Here the first map takes f to the collection with α-term given by $\mathrm{res}^U_{U_\alpha}(f)$ and the second map takes a collection $(f_\alpha)_{\alpha \in A}$ to the collection with (α, α')-term given by $\mathrm{res}^{U_\alpha}_{U_\alpha \cap U_{\alpha'}}(f_\alpha) - \mathrm{res}^{U_{\alpha'}}_{U_\alpha \cap U_{\alpha'}}(f_{\alpha'})$

Proposition 26.3.7. *If F is a sheaf then restrictions induces an isomorphism*

$$F(M) \xrightarrow{\cong} \check{H}^0_{\mathcal{U}}(M; F).$$

However, as we have seen in the example of the sheaf of locally constructions functions $\underline{\mathbb{R}}$, in general the higher Čech cohomology groups need not vanish and contain interesting information about F, M, and the open cover $\mathcal{U}$.

Chapter 27

Secondary applications of de Rham cohomology

Today we give two applications of de Rham cohomology, which establish non-triviality of geometric constructions by arguments of a “secondary” nature. That is, they exploit that when the cohomology class of a differential form vanishes it does so for a reason, namely that it is the exterior derivative of some other form.

27.1 Poincaré duals of submanifolds and intersection theory

Every closed oriented submanifold $X \subset M$ of dimension p gives rise to a linear functional

$$\begin{aligned} \rho_X \colon H^p_c(M) &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_X \omega. \end{aligned}$$

By the Poincaré duality isomorphism $(H^p_c(M))^* \cong H^{k-p}(M)$ there is a closed $(k-p)$-form $\eta_{X\subset M}$ such that $\int_M \eta_{X\subset M} \wedge \omega = i_X(\omega)$. Its cohomology class $[\eta_{X\subset M}] \in H^{k-p}(M)$ is the *Poincaré dual to X*.

Similarly, if X is compact, we can integrate any p-form over it and use the Poincaré duality isomorphism $(H^p(M))^* \cong H^{k-p}_c(M)$ to get a compactly-supported supported cohomology class $[\eta^c_{X\subset M}] \in H^{k-p}_c(M)$: the *compactly supported Poincaré dual to X*.

Our philosophy requires that there be preferred representatives of these Poincaré dual cohomology classes, and there are. Note that if we have oriented vector bundle $\pi\colon E \to B$ over an oriented manifold, the total space E admits a natural orientation using the decomposition $TE \cong \ker(d\pi) \oplus \pi^*TB$ and with this choice the Fubini theorem implies $\int_B \pi_*(-) = \int_E$.

Lemma 27.1.1. *For a d-dimensional oriented vector bundle $\pi\colon E \to X$ over a compact oriented manifold X, we have that $[\eta^c_{X\subset E}] = [\mathrm{Th}(\pi)]$.*

Proof. Since the inclusion $i\colon X \to X$ is a homotopy equivalence with homotopy inverse π, we have that $\omega - \pi^*\iota^*\omega = d\tau$. Let us now compute

$$\begin{aligned}\int_E \omega \wedge \mathrm{Th}(\pi) &= \int_E (\pi^*\iota^*\omega + d\tau) \wedge \mathrm{Th}(\pi)\\ &= \int_E \pi^*\iota^*\omega \wedge \mathrm{Th}(\pi)\\ &= \int_X \pi_*(\pi^*\iota^*\omega \wedge \mathrm{Th}(\pi))\\ &= \int_X \iota^*\omega \wedge \pi_*\mathrm{Th}(\pi)\\ &= \int_X \iota^*\omega\end{aligned}$$

where the second equation uses Stokes' theorem as $d\tau \wedge \mathrm{Th}(\pi) = d(\tau \wedge \mathrm{Th}(\pi))$ and the fourth equation the projection formula. □

This tells us that preferred representatives $\eta^c_{X\subset M}$ of the compactly-supported Poincaré dual of X is given by extension-by-zero of a representative of the Thom class $\mathrm{Th}(\pi)$ for $\pi\colon NX \to X$ the normal bundle using a tubular neighbourhood $NX \to M$.

We will now give a result summarising how two important constructions cohomology classes can be interpreted geometrically for Poincaré duals. This can be proven by appropriate choices of representatives, and for a proof and details about orientations see [BT82, p. 69]. As we make compactness hypotheses throughout, we can drop the sub- and superscripts c.

Proposition 27.1.2. *Let M be a compact smooth manifold.*

(i) If $X, Y \subset M$ are compact oriented submanifolds that intersect transversally, then we can find representatives $\eta_{X\subset M}$, $\eta_{Y\subset M}$, and $\eta_{X\cap Y\subset M}$ so that $\eta_{X\subset M} \wedge \eta_{Y\subset M} = \eta_{X\cap Y\subset M}$.

(ii) If $X \subset M$ is a compact oriented submanifold and $f\colon N \to M$ is a smooth map transverse to X then we can find representatives $\eta_{X\subset M}$ and $\eta_{f^{-1}(X)\subset N}$ so that $f^*\eta_{X\subset M} = \eta_{f^{-1}(X)\subset N}$.

If a compact submanifold $X \subset M$ is the boundary of a compact submanifold with boundary $W \subset M$ then we have that

$$\rho_X([\omega]) = \int_X \omega = \int_{\partial W} \omega = \int_W d\omega = 0$$

where the last equation uses that ω is closed. The philosophy of this lecture that is we can take advantage of *reasons* a cohomology class of a form is zero. For $\eta_{X\subset M}$ this is the existence of W. If one tries to construct a Thom class for the normal bundle of $W \subset M$ we can do so at the interior of W but needs to make modification near the boundary, and the result is a $\eta_{W\subset M}$ with the property that $d\eta_{W\subset M} = \eta_{X\subset M}$. This certifies that $[\eta_{X\subset M}] = 0$ but contains more information:

Proposition 27.1.3. *Let M be a compact smooth manifold, $W \subset M$ be a compact submanifold with boundary $\partial W = X$, and $Y \subset M$ a smooth submanifold disjoint from X and transverse to W. Then there exists choices of forms $\eta_{X\subset M}$, $\eta_{W\subset M}$, $\eta_{Y\subset M}$, and $\eta_{W\cap Y\subset M}$ so that*

(i) $\eta_{X\subset M} = d\eta_{X\subset W}$,

(ii) $\eta_{W\subset M} \wedge \eta_{Y\subset M} = \eta_{W\cap Y}$.

27.2 The Hopf invariant

Recall that $\mathbb{C}P^1$ is diffeomorphic to S^2 and was defined as the quotient $(\mathbb{C}^2 - 0)/\mathbb{C}^*$. Instead, we use only elements of norm 1 and give an equivalent definition as the quotient

$$\mathbb{C}P^1 \cong \frac{\{z \in \mathbb{C}^2 \mid |z| = 1)}{\mathrm{U}(1)},$$

or in other words, as a quotient of S^3 by a free action of S^1. The quotient map is a smooth map

$$h\colon S^3 \longrightarrow S^2$$

that we call the *Hopf fibration*. Since all fibres are circles, this gives a way of writing S^3 as a union of S^1's. We claim that this map is not null-homotopic, which we will prove by constructing an invariant of smooth maps $f\colon S^{2n-1} \to S^n$ and evaluating it for the Hopf fibration. Not only will its definition use a choice of reason that a form represents the zero cohomology class, but its evaluation will use the Poincaré dual forms of the previous section.

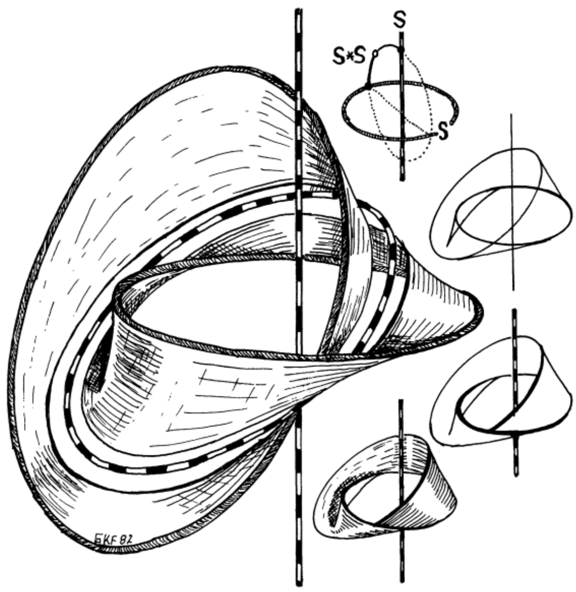


Figure 27.1 Some fibers of the Hopf fibration pictured in $\mathbb{R}^3 \subset S^3$ (from [Fra07]).

Pick an n-form $\omega \in \Omega^n(S^n)$ so that $\int_{S^n} \omega = 1$. Since $H^n(S^{2n-1}) = 0$, there exists an $(n-1)$-form ν so that $d\nu = f^*\omega$.

Definition 27.2.1. The *Hopf-invariant* of f is

$$H(f) := \int_{S^{2n-1}} \nu \wedge d\nu \in \mathbb{R}.$$

Lemma 27.2.2.

(i) $H(f)$ vanishes when n is odd.

(ii) $H(f)$ is independent of the choice of ω and ν.

(iii) $H(f)$ only depends on the homotopy class of f.

Proof. For the computations below it is helpful to observe that if a, b are p-forms then $a \wedge db = db \wedge a$ since at least one of a and db is in even degree.

For (i), note that for odd n we have

$$d(\nu \wedge \nu) = d\nu \wedge \nu + (-1)^{n-1}\nu \wedge d\nu = d\nu \wedge \nu + \nu \wedge d\nu = 2\nu \wedge d\nu$$

which integrates to zero by Stokes' theorem.

For (ii) we first prove independence of choice of ω. If we replace ω by $\omega' = \omega + d\mu$ the $(n-1)$-form $\nu' = \nu + f^*\mu$ satisfies $d\nu' = d\nu + df^*\mu = f^*\omega'$ and we compute

$$\begin{aligned}
\nu' \wedge d\nu' - \nu \wedge d\nu &= (\nu + f^*\mu) \wedge d(\nu + f^*\mu) - \nu \wedge d\nu \\
&= f^*\mu \wedge d(\nu + f^*\mu) + (\nu + f^*\mu) \wedge df^*\mu \\
&= d(f^*\mu \wedge \nu) + f^*(\mu \wedge d\mu) \\
&= d(f^*\mu \wedge \nu)
\end{aligned}$$

where the last equation uses that $\mu \wedge d\mu$ is a $(2n-1)$-form on an n-sphere and hence vanishes. By Stokes' theorem this integrates to zero. To prove independence of ν, if $d(\nu + \rho) = \omega$ then $d\rho = 0$ and we compute that

$$(\nu + \rho) \wedge d(\nu + \rho) - \nu \wedge d\nu = \rho \wedge d\nu = d(\rho \wedge \nu)$$

which integrates to zero by Stokes' theorem.

For (iii) if $H\colon S^n \times \mathbb{R} \to S^{n-1}$ is the homotopy then by homotopy invariance of de Rham cohomology we can find $\tilde{\nu} \in \Omega^n(S^{2n-1} \times \mathbb{R})$ so that $d\tilde{\nu} = H^*\omega$. By Stokes theorem we have

$$H(f_1) - H(f_0) = \int_{S^{2n-1} \times [0,1]} d(\tilde{\nu} \wedge d\tilde{\nu}) = 0,$$

since $d(\tilde{\nu} \wedge d\tilde{\nu}) = d\tilde{\nu} \wedge d\tilde{\nu} = F^*(\omega \wedge \omega) = 0$ since $\omega \wedge \omega$ is a $2n$-form on an n-sphere. $\square$

Example 27.2.3. If $f\colon S^3 \to S^2$ is null-homotopic then $H(f) = 0$. To see this, note that by (iii) we may as well assume that f is constant and then $f^*\omega = 0$ so we can take $\nu = 0$.

There is also the following useful variant of (ii), breaking the symmetry in the definition of the Hopf invariant.

Lemma 27.2.4. *If we choose ν and ν' so that $f^*\omega = d\nu$ and $f^*\omega' = d\nu'$ with $\omega' - \omega = d\mu$, then*

$$H(f) = \int_{S^{2n-1}} \nu \wedge d\nu'.$$

Proof. Consider

$$\begin{aligned}\nu \wedge d\nu' - \nu \wedge d\nu &= \nu \wedge f^* d\mu \\ &= (-1)^{n-1} d(\nu \wedge f^*\mu) - (-1)^{n-1} d\nu \wedge f^*\mu \\ &= (-1)^{n-1} d(\nu \wedge f^*\mu) - (-1)^{n-1} f^*(\omega \wedge \mu) \\ &= (-1)^{n-1} d(\nu \wedge f^*\mu)\end{aligned}$$

where the last equality uses that $\omega \wedge \mu$ is a $(2n-1)$-form on an n-spheres and hence vanishes, and the last term integrates to zero by Stokes. □

It is possible to prove the following by a direct computation of integrals [BT82, p. 235–238] but it is more easily done using Poincaré duals:

Theorem 27.2.5. *If $f\colon S^3 \to S^2$ is the Hopf fibration, then $H(f) = 1$.*

Proof. Note since $[\omega] = [\eta_{p \subset S^2}]$ we may assume $f^*\omega = \eta_{f^{-1}(p) \subset S^3}$ for a fibre $f^{-1}(p) \cong S^1$ of the Hopf fibration. This fibre bounds a 2-disc D in S^3 so we may assume $\eta_{f^{-1}(p) \subset S^3} = d\eta_{D \subset S^3}$ and can take $\nu = \eta_{D \subset S^3}$. Doing the same for a nearby point p' we can take $d\nu' = \eta_{f^{-1}(p) \subset S^3}$ and noting that this is transverse to D intersecting it in a single point $* \in S^3$, with positive orientation had we kept track of orientations, we have

$$H(f) = \int_{S^3} \eta_{D \subset S^3} \wedge \eta_{f^{-1}(p') \subset S^3} = \int_{S^3} \eta_{* \subset S^3} = 1.$$ □

By precomposing or postcomposing with self-maps of S^3 or S^2 of arbitrary degree, we see that the Hopf invariant takes all integer values and we get:

Corollary 27.2.6. *There are infinitely many homotopy classes of maps $S^3 \to S^2$.*

27.3 Massey products

A second example of secondary invariants are Massey products. Suppose that we have closed forms ω, ν, μ on M so that $[\omega \wedge \nu] = 0 = [\nu \wedge \mu]$. That is, we have $\omega \wedge \nu = d\alpha$ and $\nu \wedge \mu = d\beta$. If we take the element

$$x = \omega \wedge \beta + (-1)^{|\omega|} \alpha \wedge \mu$$

then it satisfies

$$dx = (-1)^{|\omega|} \omega \wedge \nu \wedge \mu + (-1)^{|\omega|} \omega \wedge \nu \wedge \mu = 0$$

and hence represents a cohomology class. It is independent of choices once we pass to a certain quotient:

Lemma 27.3.1. *The class*

$$[x] \in \frac{H^{|\omega|+|\nu|+|\mu|-1}(M)}{[\omega] \wedge H^{|\nu|+|\mu|-1}(M) + H^{|\omega|+|\nu|-1}(M) \wedge [\mu]}$$

is independent of choice of α and β, as well as of representatives ω, ν, μ of the cohomology class $[\omega], [\nu], [\mu]$.

Proof. We first show that it is independent of the choice of β, and the proof that is independent of the choice of α is analogous. If $d\beta' = \nu \wedge \nu = d\beta$ then $d(\beta' - \beta) = 0$ and we see that

$$\omega \wedge \beta' + (-1)^{|\omega|}\alpha \wedge \mu - \omega \wedge \beta - (-1)^{|\omega|}\alpha \wedge \mu = \omega \wedge (\beta' - \beta)$$

is a representative of $[\omega] \wedge [\beta' - \beta]$ and goes to zero in the quotient.

We next prove that it is independent of the representative ω of $[\omega]$, and the proof that is independent of the choices of ν and μ is analogous. If ω and ω' both represent $[\omega]$ then $\omega' - \omega = d\rho$ and we may take $\alpha' = \alpha + \rho \wedge \nu$ and we have

$$\begin{aligned}\omega' \wedge \beta + (-1)^{|\omega|}\alpha \wedge \mu - \omega \wedge \beta - (-1)^{|\omega|}\alpha' \wedge \mu &= (\omega - \omega') \wedge \beta + (-1)^{|\omega|}(\alpha - \alpha') \wedge \mu \\ &= d\rho \wedge \beta + (-1)^{|\omega|}(\alpha - \alpha') \wedge \mu \\ &= d\rho \wedge \beta - (-1)|\omega|\rho \wedge \nu \wedge \mu \\ &= d(\rho \wedge \beta). \square\end{aligned}$$

Definition 27.3.2. We call $[x]$ the *Massey product* of $[\omega], [\nu], [\mu]$ and denote it by

$$\langle [\omega], [\nu], [\mu] \rangle \in \frac{H^{|\omega|+|\nu|+|\mu|-1}(M)}{[\omega] \wedge H^{|\nu|+|\mu|-1}(M) + H^{|\omega|+|\nu|-1}(M) \wedge [\mu]}.$$

The following example is due to Morita [?, Example 3.24], of a non-trivial Massey product in a so-called nilmanifold:

Example 27.3.3. We consider the Lie group

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\} \subset \mathrm{GL}_3(\mathbb{R}).$$

This is easily seen to be a smooth manifold of dimension 3, and it has a subgroup $\Gamma \subset N$ of those matrices where x, y, z are integers. This acts freely and properly on N, so the quotient $M = N/\Gamma$ is also a smooth manifold of dimension 3. The forms dx, dy and $dy + xdz$ on N are Γ-invariant so descend to unique 1-forms α, β, and γ on M. It turns out that $H^1(M)$ is 2-dimensional generated by $[\alpha]$ and $[\beta]$ and $H^2(M)$ is also 2-dimensional generated by $[\alpha \wedge \gamma]$ and $[\alpha \wedge \gamma]$.

It is of course true that $\alpha \wedge \alpha = 0$ and we have that $\alpha \wedge \beta = d\gamma$. Thus we can form the Massey product

$$\langle [\alpha], [\alpha], [\beta] \rangle \in \frac{H^2(M)}{[\alpha] \wedge H^1(M) + H^1(M) \wedge [\beta]} = H^2(M)$$

and it is represented by $\alpha \wedge \gamma$ so non-zero.

Chapter 28

Flows along vector fields

Even though we are now familiar with de Rham cohomology, a question remains: what is its geometric significance? For the remainder of these notes, our goal is to connect Morse theory to de Rham cohomology. Today we start the technical preparations. This material can be found in Section 1.4 of [Wal16].

28.1 Flows along vector fields

When we do Morse theory on a smooth manifold M in the next lectures, we will deform subsets of M by flowing them along the gradient vector field of a Morse function $f\colon M \to \mathbb{R}$ (to define the gradient we will need to pick a Riemannian metric). Thus, we have to define flows along vector fields on manifolds: as usual, we take a known result on open subsets of $\mathbb{R}^k$ and extend it to k-dimensional manifolds using charts.

28.1.1 Flows on $\mathbb{R}^k$

The result we use is the existence and uniqueness theorem for solutions to ordinary differential equations, cf. [Wal16, Theorem 1.4.1]:

Theorem 28.1.1. *Let $U \subset \mathbb{R}^k$ be open, $K \subset U$ be compact, and $\mathcal{X}$ a smooth vector field on U. Then there exists an $\epsilon > 0$, an open neighbourhood $U' \subset U$ of K, and a unique smooth map $\Psi\colon U' \times (-\epsilon, \epsilon) \to U$ such that*

$$\frac{d}{dt}\Psi(x,t) = \mathcal{X}(\Psi(x,t)) \qquad \textit{and} \qquad \Psi(x,0) = x.$$

Let us restate this using the following notion:

Definition 28.1.2. An *integral curve* for $\mathcal{X}$ through x, is a smooth map $\gamma\colon (-\epsilon,\epsilon) \to U$ such that $\gamma(0) = x$ and $\frac{d}{dt}\gamma(t) = \mathcal{X}(\gamma(t))$.

Theorem 28.1.1 says that integral curves exist, are unique, and depend smoothly on the initial condition. For $t \in (-\epsilon,\epsilon)$, let us denote by ψ_t the map $x \mapsto \Psi(x,t)$. We call Ψ the *flow* and ψ_t the *flow for time t*, and it has the following properties:

Proposition 28.1.3. $\psi_0 = \mathrm{id}$ *and* $\psi_t(\psi_s(x)) = \psi_{s+t}(x)$ *whenever both are defined.*

Proof. The first property is clear. The second property uses that Ψ is unique. The map $t \mapsto \Psi(x, s+t)$ at $t = 0$ is equal to $g(x, s)$ and has derivative $\frac{d}{dt}\Psi(x, s+t) = \mathcal{X}(\Psi(x, s+t))$. That is, it has the properties uniquely defining $\Psi(x', t)$ with $x' = \Psi(x, s)$. Thus we see that

$$\psi_{s+t}(x) = \Psi(x, s+t) = \Psi(\Psi(x, s), t) = \psi_t(\psi_s(x)). \qquad \square$$

You can recover the vector field $\mathcal{X}$ from the flow Ψ as the derivative of $\Psi(-, t)$ with respect to t at $t = 0$.

28.1.2 Flows on manifolds

To extend these results to smooth manifolds, we study the behaviour of solutions to ordinary differential equations under diffeomorphisms. Given a diffeomorphism $\phi\colon \mathbb{R}^k \supset U \to V \subset \mathbb{R}^k$, we can push forward $\mathcal{X}$ along ϕ to get a vector field $\phi_*\mathcal{X}$ on V. In fact, the pushforward of vector fields is defined on arbitrary manifolds, and is given by using the applying derivative of the diffeomorphism to the vector field:

Definition 28.1.4. If $\varphi\colon M \to N$ is a diffeomorphism and $\mathcal{X}$ is a vector field on M, then the *pushforward of* $\mathcal{X}$ *along* φ is given by

$$\varphi_*\mathcal{X}(p) := d_{\varphi^{-1}(p)}\varphi\left[\mathcal{X}(\varphi^{-1}(p))\right].$$

For open subsets of Euclidean space the derivative is given by total derivative and we have

$$\phi_*\mathcal{X} = D_{\phi^{-1}(x)}\phi\left[\mathcal{X}(\phi^{-1}(x))\right].$$

On the one hand we can apply Theorem 28.1.1 to $\phi_*\mathcal{X}$ on V using the compact $K' := \phi(K)$. The result is a solution $\Psi'\colon V' \times (-\epsilon', \epsilon') \to V$ to the differential equation

$$\frac{d}{dt}\Psi'(x', t) = \phi_*\mathcal{X}(\Psi'(x', t)) \qquad \text{and} \qquad \Psi'(x', 0) = x'. \tag{28.1}$$

On the other hand we can transport the solution Ψ to $\mathcal{X}$ using ϕ:

$$\begin{aligned} \Psi''\colon \phi(U') \times (-\epsilon, \epsilon) &\longrightarrow V \\ (x', t) &\longmapsto \phi(\Psi(\phi^{-1}(x'), t)). \end{aligned}$$

I claim that this is a solution to (28.1). To prove this, observe it satisfies

$$\Psi''(x', 0) = \phi(\Psi(\phi^{-1}(x'), 0)) = \phi(\phi^{-1}(x')) = x',$$

and that we can use the chain rule to deduce that

$$\begin{aligned} \frac{d}{dt}\Psi''(x', t) &= D_{\Psi(\phi^{-1}(x'),t)}\phi\left[\frac{d}{dt}\Psi(\phi^{-1}(x'), t)\right] \\ &= D_{\phi^{-1}(\Psi''(x',t))}\phi\left[\mathcal{X}(\Psi(\phi^{-1}(x'), t))\right] \\ &= (\phi_*\mathcal{X})(\Psi''(x', t)). \end{aligned}$$

By uniqueness, any other solution of (28.1) has to coincide on $\Psi''(x,t)$ on the intersection of their domain of definition: hence $\Psi'' = \Psi'$ on $(\phi(U') \cap V') \times (-\min(\epsilon,\epsilon'), \min(\epsilon,\epsilon'))$.

We will use this result to extend the technique of flowing along vector fields to manifolds.

Theorem 28.1.5. *Let M be a smooth manifold and $\mathcal{X}$ be a vector field on M. Then there exists a smooth map $\eta\colon M \to \mathbb{R}_{>0}$ and a unique smooth map $\Psi\colon \{(p,t) \in M \times \mathbb{R} \mid |t| < \eta(p)\} \to M$ such that*

$$\frac{d}{dt}\Psi(p,t) = \mathcal{X}(\Psi(p,t)) \qquad \textit{and} \qquad \Psi(p,0) = p. \tag{28.2}$$

Proof. We can find a collection of charts $\phi_\alpha\colon \mathbb{R}^k \supset U_\alpha \to V_\alpha \subset M$ and compact subsets $K_\alpha \subset V_\alpha$ such that the K_α cover M.

Every point $p \in M$ lies in some compact subset K_α of M. We can push forward the restriction $\mathcal{X}|_{V_\alpha}$ to U_α along ϕ_α^{-1} and apply Theorem 28.1.1 to the resulting vector field $(\phi_\alpha^{-1})_*\mathcal{X}$. This gives us a smooth map $\tilde{\Psi}_\alpha\colon U'_\alpha \times (-\epsilon_\alpha, \epsilon_\alpha) \to U_\alpha$ with U'_α an open neighbourhood of $\phi_\alpha^{-1}(K_\alpha)$. As above, we get a solution Ψ_α to (28.2) on an open neighbourhood of $K_\alpha \times (-\epsilon_\alpha, \epsilon_\alpha)$, by setting its value on $(p,t) \in K_\alpha \times (-\epsilon_\alpha, \epsilon_\alpha)$ to be

$$\Psi_\alpha(p,t) \coloneqq \phi_\alpha(\tilde{\Psi}_\alpha(\phi_\alpha^{-1}(p), t).$$

We must check that combining these local solutions to (28.2) give rise to a well-defined smooth map Ψ. That is, if $p \in K_\alpha \cap K_\beta$, then we should have

$$\phi_\alpha(\tilde{\Psi}_\alpha(\phi_\alpha^{-1}(p), t) = \phi'_\beta(\tilde{\Psi}_\beta((\phi'_\beta)^{-1}(p), t)$$

as long as t is small enough so that both are defined. This is guaranteed by the previous discussion applied to the diffeomorphism $(\phi_\beta)^{-1} \circ \phi_\alpha\colon \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \to (\phi_\beta)^{-1}(V_\alpha \cap V_\beta)$; pushing forward the vector field $(\phi_\alpha^{-1})_*\mathcal{X}$ along this diffeomorphism gives $(\phi_\beta^{-1})_*\mathcal{X}$.

The result is a solution to (28.2) defined on an open neighbourhood V of $M \times \{0\}$ in $M \times \mathbb{R}$. Such an open subset always contains one of the type mentioned in the theorem. □

Remark 28.1.6. This proof is one the places where it is important that manifolds are Hausdorff: on the line with doubled origin the flow along $\frac{\partial}{\partial x}$ exists but is not unique (you have to decide which of the origins to go into). This Hausdorffness assumption is hidden in the proof: it is used to see that $K_\alpha \cap K_\beta$ is compact.

As in the local case, we can define $\psi_t(p) = \Psi(p,t)$ for $(p,t) \in V$. This satisfies $\phi_0(p) = p$ and $\psi_t(\psi_s(p)) = \psi_{s+t}(p)$ as long as both are defined, and one can recover $\mathcal{X}$ from the flow by taking the derivative of $\Psi(-,t)$ with respect to t at $t = 0$.

What can we say about the domain of definition? By uniqueness any two solutions to (28.2) agree on the overlap of their domain of definitions, so by combining these we can extend the domain. In particular, there is a solution with maximal domain of definition. However, even for a solution with maximal domain, $t \mapsto \Psi(p,t)$ might still only be defined on some proper open interval $(a_p, b_p) \subset \mathbb{R}$ with $a_p < 0$ and $b_p > 0$:

Example 28.1.7. Let $M = \mathbb{R}\backslash\{0\}$ and $\mathcal{X} = \frac{\partial}{\partial x}$. Then the maximal domain of definition of Ψ is given by those $(x,t) \in \mathbb{R} \times \mathbb{R}$ such that $t > x$ if $x < 0$, $t < x$ if $t < x$ if $x > 0$.

However, this can only occur if the integral curve through p leaves all compact subsets of M eventually. Lemma 1.4.3 of [Wal16] says:

Lemma 28.1.8. *Suppose Ψ has maximal domain and fix $p \in M$. Either $b_p = \infty$ or the map $\Psi(p,-)\colon [0,b_p) \to M$ is proper. Similarly, either $a_p = -\infty$ or the map $\Psi(p,-)\colon (a_p,0] \to M$ is proper.*

Corollary 28.1.9. *Suppose M is compact. If a solution to* (28.2) *has maximal domain then its domain is $M \times \mathbb{R}$.*

Proof. As M is compact, no map $[0,b_p) \to M$ or $(a_p,0] \to M$ is proper. □

Remark 28.1.10. When M is compact, this corollary implies there is a one-to-one correspondence between 1-parameter groups of diffeomorphisms and smooth vector fields.

There are other conditions under which the maximal domain is all of $M \times \mathbb{R}$, e.g. if $\mathcal{X}$ is compactly-supported or more generally, if $\mathcal{X}$ coincides outside of a compact subset with a vector field $\mathcal{Y}$ whose maximal domain is $M \times \mathbb{R}$.

28.2 Isotopy extension

We will now give the first of several important applications of flows along vector fields, a very important geometric tool called *isotopy extension.*

28.2.1 The isotopy extension theorem

It is based on the following idea: if you imagine your smooth manifold M as being made from a stretchy fabric, then you can use your finger to move one point $p \in M$ to some other point $p' \in M$ and deform the rest of the manifold along to produce a diffeomorphism $M \to M$ which moves p to p'.

In other words, imagining M as being made out of a stretchy fabric suggests than any isotopy of embeddings $* \to M$ (starting at the map with value p and ending at the map with value p') can be extended to an isotopy of diffeomorphisms $M \to M$. An isotopy of diffeomorphisms is also called an *ambient isotopy*, suggesting the following interpretation: you do not just move the objects in question but also their surrounding environment.

The isotopy extension theorem says that an isotopy extends to an ambient isotopy under mild assumptions.

Theorem 28.2.1 (Isotopy extension)**.** *Suppose that M and X smooth manifolds without boundary, and that X is compact. Then for any isotopy of embeddings*

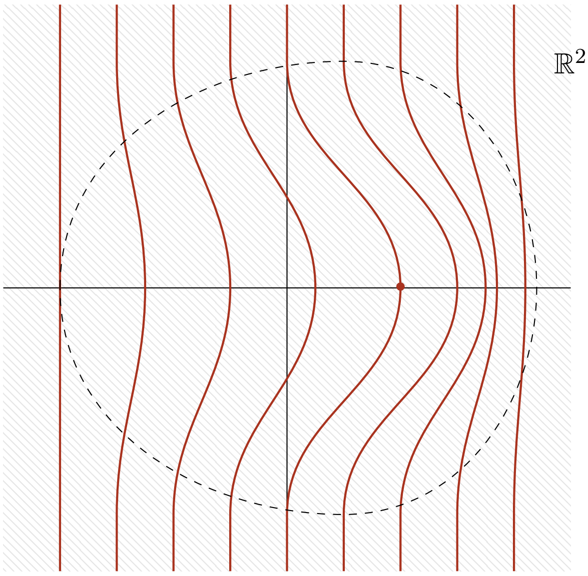


Figure 28.1 The end result of pushing the origin to the red point, depicted by its effect on vertical lines in $\mathbb{R}^2$. The dashed line gives the boundary of the support.

$e_t\colon X \times [0,1] \to M$ *can be extended to an isotopy of diffeomorphisms, in the following sense: there exists a family of diffeomorphisms* $\phi_t\colon M \times [0,1] \to M$ *satisfying* $\phi_0 = \mathrm{id}$ *and* $\phi_t \circ e_0 = e_t$*. Furthermore, each* ϕ_t *will be compactly-supported (that is, equal to the identity outside a compact subset).*

Proof. Let us define $e\colon X \times [0,1] \to M \times \mathbb{R}$ by $e(p,t) = e_t(p)$. The smooth vector field on $X \times [0,1]$ given by $\frac{\partial}{\partial t}$ can be pushed forward along the embedding e to obtain a vector field $\mathcal{X}$ on $e(X \times [0,1]) \subset M \times [0,1]$. Suppose we could extend this to a vector field $\mathcal{X}'$ on all of $M \times \mathbb{R}$. Then I claim that if we flow along $\mathcal{X}'$ for time t with initial condition $(e_0(p), 0)$, we end up at $(e_t(p), t)$. To see this, we must prove that

$$t \longmapsto (e_t(p), t)$$

is an integral curve for $\mathcal{X}'$. To see this, takes its derivative with respect to t and apply the chain rule

$$\frac{d}{dt}(e_t(p), t) = d_{(p,t)}e \left[\frac{d}{dt}(t \mapsto (p,t))\right] = e_* \left[\frac{\partial}{\partial t}(p,t)\right] = \mathcal{X}'(e(p,t)).$$

In other words, flowing e_0 with image in $M \times \{0\}$ along $\mathcal{X}'$ for time t produces e_t with image in $M \times \{t\}$. We can try to produce ϕ_t by flowing the identity map of $M \times \{0\}$ along $\mathcal{X}$ for time t. There are two problems:

(i) the flow may not exist,

(ii) it is not necessarily the case that the flow sends $M \times \{0\}$ to $M \times \{t\}$.

Problem (ii) is solved by extending $\mathcal{X}$ not just to any smooth vector field $\mathcal{X}'$ on $M \times \mathbb{R}$, but one that projects to $\frac{\partial}{\partial t}$ under $d\pi$ for $\pi\colon M \times \mathbb{R} \to \mathbb{R}$. If so, we get the differential equation

$$\frac{d}{dt}\left(\pi \circ \psi_t(p,s)\right) = d\pi \circ \mathcal{X}'(\psi_t(p,s)) = \frac{\partial}{\partial t},$$

and the initial condition $\pi \circ \psi_0(p,s) = s$ guarantees that $\pi \circ \psi_t(p,s) = s + t$.

If we make sure that $\mathcal{X}'$ is equal to $\frac{\partial}{\partial t}$ outside of a compact set, this will solve problem (i). It guarantees that the flow exists, because $\mathcal{X}'$ coincides outside of a compact set with a vector field whose maximal domain of solution is all of $M \times \mathbb{R} \times \mathbb{R}$. Having imposed theses conditions, we can thus prove the theorem by taking

$$\begin{aligned} \phi\colon M \times [0,1] &\longrightarrow M \times [0,1] \\ (p,t) &\longmapsto \Psi((p,0),t), \end{aligned}$$

or in other words, $\phi_t(p) = \psi_t(p,0)$.

So it remains to construct an extension $\mathcal{X}'$ with the desired properties. Firstly, it suffices to construct a smooth vector field $\mathcal{X}'$ which

(a) coincides with $\mathcal{X}$ on $e(X \times [0,1])$,

(b) coincides with $\frac{\partial}{\partial t}$ outside a compact subset of $M \times \mathbb{R}$,

(c) satisfies the property $d\pi \circ \mathcal{X}'$ is a positive multiple of $\frac{\partial}{\partial t}$ everywhere.

We may then afterwards modify $\mathcal{X}'$ by scaling it with smooth function that is 1 on $X \times \mathbb{R}$, to get that $d\pi \circ \mathcal{X}' = \frac{\partial}{\partial t}$ on all of $M \times \mathbb{R}$.

Since X is compact, we may find a finite collection of charts $\phi_i\colon \mathbb{R}^k \supset U_i \to V_i \subset M \times \mathbb{R}$ covering the image of e and satisfy $\phi_i^{-1}(V_i \cap e(X \times [0,1])) = U_i \cap (\mathbb{R}^{m-1} \times [0,\infty) \times \{0\})$. Let $\mathcal{X}_i'$ be the vector field on V_i given as follows:

Step (i): first extend $(\phi_i)_*(\frac{\partial}{\partial t})$ on $U_i \cap (\mathbb{R}^{m-1} \times [0,\infty) \times \{0\})$ to $U_i \cap (\mathbb{R}^m \times \{0\})$,

Step (ii): then extend it in constant manner to the remaining $(k-m)$ coordinate directions of U_i,

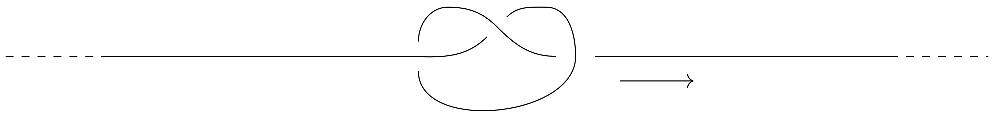
Step (iii): apply $(\phi_i^{-1})_*$.

This extends $\mathcal{X}|_{V_i}$, so in particular has the property that $d\pi \circ \mathcal{X}_i' = \frac{\partial}{\partial t}$ on $V_i \cap e(X \times [0,1])$. Hence by possibly shrinking V_i to a smaller open neighborhood of $V_i \cap e(X \times [0,1])$, we may assume that $\pi_*(\mathcal{X}_i')$ is a positive multiple of $\frac{\partial}{\partial t}$.

Let V_0 be an open subset of $M \times [0,1]$ satisfying $V_0 \cap e(X \times [0,1]) = \varnothing$ and $V_0 \cup \bigcup_{i=1}^k V_i = M \times [0,1]$, and let η_i be smooth partition of unity subordinate to this open cover. The desired vector field is

$$\mathcal{X}' := \eta_0 \cdot \frac{\partial}{\partial t} + \sum_{i=1}^k \eta_i \cdot \mathcal{X}_i'.$$

By construction this extends $\mathcal{X}$ and the condition that $d\pi \circ \mathcal{X}'$ is a multiple of $\frac{\partial}{\partial t}$ by a positive smooth function is preserved by taking convex linear combinations such as those that appear when using partitions of unity. $\square$



An embedding of $\mathbb{R}$ into $\mathbb{R}^3$ given by knot X centered at the origin for $t = 0$ moving rightwards to ∞ as t increases.

Figure 28.2 A family of embeddings to which isotopy extension does not apply. It does not satisfy the assumption that X is compact.

28.2.2 Transitivity of diffeomorphisms

We have previously asserted that there exists a diffeomorphism of $\mathbb{R}^n$ mapping the origin to any specified point $x \in \mathbb{R}^n$. Let us use isotopy extension to generalize this to all connected manifolds:

Corollary 28.2.2. *Suppose that M is a connected manifold and $p, p' \in M$, then there exists a compactly-supported diffeomorphism $\varphi\colon M \to M$ such that $\varphi(p) = \varphi(p')$.*

In fact, the proof will give a stronger result: we can find such a φ which is isotopic to the identity.

Proof. Since M is connected, there exists a path γ from p to p'. Defining

$$\begin{aligned} e\colon * \times [0,1] &\longrightarrow M \\ (*, t) &\longmapsto \gamma(t), \end{aligned}$$

this can be interpreted as an isotopy of embeddings from the embedding

$$\begin{aligned} e_0\colon * &\longrightarrow M \\ * &\longmapsto p \end{aligned}$$

to the embedding

$$\begin{aligned} e_1\colon * &\longrightarrow M \\ * &\longmapsto p'. \end{aligned}$$

Applying the isotopy extension theorem to e, we find an isotopy $\phi_t\colon M \times [0,1] \to M$ such that $\phi_0 = \mathrm{id}$ and $\phi_t \circ e_0 = e_t$. Then ϕ_1 is the desired diffeomorphism. $\square$

28.2.3 Knot complements

It follows from Corollary 28.2.2 that $M \backslash p$ and $M \backslash p'$ are diffeomorphic; the restriction of φ gives this diffeomorphism. This can be generalized as follows.

Recall that a *knot* is an embedding $e\colon S^1 \to \mathbb{R}^3$ up to isotopy. One might think of trying to distinguish a knot by its complement $\mathbb{R}^3 \backslash e(S^1)$. However, it is not obviously clear this is well-defined, because its diffeomorphism type may depend on the choice of e within the isotopy class. However, the isotopy extension theorem tells us that for any two representatives $e, e' \colon S^1 \to \mathbb{R}^3$ of a knot, there exists a diffeomorphism $\varphi \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi \circ e = e'$. This restrict to a diffeomorphism

$$\varphi|_{\mathbb{R}^3 \backslash e(S^1)} \colon \mathbb{R}^3 \backslash e(S^1) \longrightarrow \mathbb{R}^3 \backslash e'(S^1).$$

28.3 Manifold bundles and the Ehresmann fibration theorem

28.3.1 Manifold bundles

The data of a smooth vector bundle in particular is a smooth map $p \colon E \to X$ whose fibers are diffeomorphic to $\mathbb{R}^k$. It must be locally trivial, in the sense that each point $x \in X$ admits an open neighbourhood V and a commutative diagram

$$\begin{tikzcd} \pi^{-1}(V) \arrow[r, "\cong"] \arrow[d, "\pi"] & V \times \mathbb{R}^k \arrow[d, "\pi_2"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

the horizontal maps are diffeomorphisms. There is nothing special about $\mathbb{R}^k$ here, and we can replace it with any other smooth manifold M:

Definition 28.3.1. Suppose that either $\partial M = \varnothing$ or $\partial X = \varnothing$. A *smooth manifold bundle with fiber M* is a smooth map $\pi \colon E \to X$ such that for each point $x \in X$ there is an open neighbourhood V and a commutative diagram

$$\begin{tikzcd} \pi^{-1}(V) \arrow[r, "\cong"] \arrow[d, "\pi"] & V \times M \arrow[d, "\pi_1"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

with horizontal maps diffeomorphisms.

Usually both ∂M and ∂X will be empty. We will denote the fiber $p^{-1}(x)$ by E_x; by definition it is diffeomorphic to M.

Example 28.3.2. There is always a trivial manifold bundle $\pi_1 \colon X \times M \to X$.

Example 28.3.3. Suppose that $\partial M \neq \varnothing$ (hence we assume $\partial X = \varnothing$), then $p|_{\partial E} \colon \partial E \to X$ is a smooth manifold bundle with fiber ∂M. Indeed, the local trivializations in Definition 28.3.1 restrict to local trivializations

$$\begin{tikzcd} \partial\pi^{-1}(V) \arrow[r, "\cong"] \arrow[d, "\pi"] & V \times \partial M \arrow[d, "\pi_2"] \\ V \arrow[r, equal] & V. \end{tikzcd}$$

Example 28.3.4. If a compact Lie group G acts freely and smoothly on M, then $M \to M/G$ is a smooth manifold bundle with fiber G.

28.3.2 The Ehresmann fibration theorem

Theorem 28.3.5 (Ehresmann fibration theorem)**.** *A proper submersion $\pi\colon E \to X$ is a manifold bundle.*

Proof. It remains to check that $p\colon E \to X$ is locally trivial. That is, we need to find for each point $x \in X$ a local trivialization: an open neighbourhood U of x and a commutative diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[d, "\pi"] & U \times M \arrow[d, "\pi_1"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

with horizontal maps diffeomorphisms. By restricting to a chart in X, we thus may assume without loss of generality that $X = \mathbb{R}^k$ and x is the origin.

By induction over k, it suffices to prove that a proper submersion $p\colon E \to \mathbb{R}^k$ whose restriction to $\mathbb{R}^{k-1} \times \{0\}$ has a trivialization, has a local trivialization near the origin. To do so, it suffices to find a commutative diagram

$$\begin{tikzcd} \pi^{-1}(\mathbb{R}^{k-1} \times \{0\}) \times \mathbb{R} \arrow[r, "\cong", "G"'] \arrow[d, "\pi \times \mathrm{id}"] & \pi^{-1}(\mathbb{R}^{k-1} \times \mathbb{R}) = E \arrow[d, "\pi"] \\ \mathbb{R}^{k-1} \times \mathbb{R} \arrow[r, equal] & \mathbb{R}^{k-1} \times \mathbb{R} \end{tikzcd}$$

with horizontal maps diffeomorphisms.

To do so, we use a vector field $\mathcal{X}$ on E such that $dp \circ \mathcal{X} = \frac{\partial}{\partial x_k}$. Such a vector field can clearly be constructed locally using charts provided by the submersion theorem, and these can be combined using a partition of unity as in the proof of Theorem 28.2.1. Now we apply Theorem 28.1.5 to $\mathcal{X}$ and consider the maximal domain of each integral curve. For $p \in E$ the maximal domain of the integral curve through p either (i) is $\mathbb{R}$, (ii) the maximal integral curve gives a proper map $\gamma_p\colon (a_p, 0]$ or $\gamma_p\colon [0, b_p) \to M$ with $a_p \neq -\infty$ or $b_p \neq \infty$. We rule out case (ii): the composition $\pi \circ \gamma_p$ is proper since π and γ_p are and by uniqueness of solutions to ordinary differential equations given by $t \mapsto \pi(p) + t \cdot e_k$. But this map is not proper unless both $a_p = -\infty$ or $b_p = \infty$. Thus the flow is defined on all of $M \times \mathbb{R}$.

In terms of this, the map G is given by

$$\begin{aligned} \pi^{-1}(\mathbb{R}^{k-1} \times \{0\}) \times \mathbb{R} &\longrightarrow \pi^{-1}(\mathbb{R}^{k-1} \times \mathbb{R}) \\ (p, t) &\longmapsto g(p, t) \end{aligned}$$

with inverse given by mapping $p' \in E$ to $(g(p', -\mathrm{pr}_k \circ \pi(p')), \mathrm{pr}_k \circ \pi(p'))$: the fact that $d\pi \circ \mathcal{X} = \frac{\partial}{\partial x_k}$ guarantees this is well-defined and that the composition of G with π is equal to $\pi \times \mathrm{id}$. $\square$

Using the result that quotients of free smooth actions of compact Lie groups are submersions, this implies the following:

Corollary 28.3.6. *If a compact Lie group G acts freely and smoothly on M, then the quotient map $M \to M/G$ is a manifold bundle with fibers diffeomorphic to G.*

28.4 Problems

Problem 61. Let G be a compact connected Lie group.

(a) Show that there is an isomorphism between the tangent space T_eG and the vector space left-invariant vector fields on G.

(b) For $X \in T_eG$, let φ_t^X be the flow generated by the left-invariant vector field corresponding to X. Prove that its maximal domain is $\mathbb{R}$.

Chapter 29

First fundamental theorem of Morse theory

In this lecture, we discuss that part of Morse theory which does not involve critical points. We define Morse functions, prove they exist, and show that if $[a, b] \subset \mathbb{R}$ contains no critical points of f, then $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times [a, b]$. This can be found in Section 1.7 of [GP10], Section 5.1 of [Wal16], and Section 3 of [Mil63].

29.1 Morse functions

Recall that for a smooth function $f : M \to \mathbb{R}$, a point $p \in M$ so that $d_p f$ is not surjective is called a *critical point*. Given a critical point and local coordinates $(x_1, \ldots, x_k)$, one can define the *Hessian*. For simplicity, suppose that p is the origin in these local coordinates, then we have a $(k \times k)$-matrix with (i, j)th entry given by

$$\mathrm{Hess}_0(f)_{ij} := \frac{\partial^2 f}{\partial x_i \partial x_j}(0).$$

Remark 29.1.1. By Taylor's theorem, in these local coordinates f is near the origin given by

$$f(x) = f(0) + \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 f}{\partial x_i \partial x_j}(0)\, x_i x_j + O(x^3).$$

We say that p is a *non-degenerate critical point* if the Hessian matrix as described above is invertible. Though the Hessian itself depends on a choice of coordinates, it being invertible is well-defined, by the following lemma which is an easy consequence of the chain rule:

Lemma 29.1.2. *If $\phi : \mathbb{R}^k \supset U \to U' \subset \mathbb{R}^k$ is a diffeomorphism such that $\phi(0) = 0$. Then the origin is a non-degenerate critical point $f : U' \to \mathbb{R}$ if and only if it is a non-degenerate critical point of $f \circ \phi$.*

Definition 29.1.3. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if all its critical points are non-degenerate.

Example 29.1.4. It follows from the expression in Remark 29.1.1 that non-degenerate critical points are isolated. In particular, a Morse function on a compact manifold only has finitely many critical points.

29.1.1 Existence of Morse functions

Morse functions are generic among smooth maps $f\colon M \to \mathbb{R}$. This follows from the following theorem, which depends on a choice of embedding $e\colon M \hookrightarrow \mathbb{R}^N$ (this exists by Whitney embedding theorem). Let $e_1, \dots, e_N\colon M \to \mathbb{R}$ denote the coordinates of e.

Theorem 29.1.5. *For a dense set of* $(a_1, \dots, a_N) \in \mathbb{R}^N$*, the smooth map*

$$\begin{aligned} f_a\colon M &\longrightarrow \mathbb{R} \\ p &\longmapsto f(p) + a_1 e_1(p) + \cdots + a_N e_N(p) \end{aligned}$$

is a Morse function.

Proof. We shall denote the map in the statement of the theorem as f_a.

We first consider the local situation; suppose $U \subset \mathbb{R}^k$ is an open subset and $g\colon U \to \mathbb{R}$ is a smooth function. Then we claim that for outside of a set of $b \in \mathbb{R}^k$ of measure zero, the map

$$\begin{aligned} g_b\colon U &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_k) &\longmapsto g(x_1, \dots, x_k) + b_1 x_1 + \cdots + b_k x_k \end{aligned}$$

is a Morse function. To do so, we observe that p is a critical point of g_b if and only if $D_p g = -b$.

Since we working on $\mathbb{R}^k$, the Hessian is well-defined even at points which are not critical point. Thus it makes sense to say that g and g_b have the same Hessians; this is true because g_a is obtained by adding a linear perturbation to g. We next consider the function

$$\begin{aligned} G\colon U &\longrightarrow \mathbb{R}^k \\ (x_1, \dots, x_k) &\longmapsto \left(\frac{\partial g}{\partial x_1}(x_1, \dots, x_k), \dots, \frac{\partial g}{\partial x_k}(x_1, \dots, x_k) \right) \end{aligned}$$

because b is a critical point of G if and only if the Hessian of g (or equivalently g_b) at p is non-degenerate. Thus g_b is Morse if and only if b is not a critical value of G. By Sard's theorem these critical values have measure zero.

Having established this local statement, we use a prove the global one. To do so, we find a countably open cover $\{U_\alpha\}$ of M such that for each U_α there exist k integers $i_1, \dots, i_k$ in $\{1, \dots, N\}$ such that coordinate functions $e_{i_1}, \dots, e_{i_k}\colon U_\alpha \to \mathbb{R}$ give local coordinates on U_α. Without loss of generality we have $i_j = j$ for $j \in \{1, \dots, k\}$ and we can use $e_1, \dots, e_k$ as local coordinates $x_1, \dots, x_k$ on U_α. Then for each $c_{k+1}, \dots, c_N \in \mathbb{R}$ we consider the smooth function

$$\begin{aligned} f^c\colon U_\alpha &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_k) &\longmapsto f(x_1, \dots, x_k) + c_{k+1}\, e_{k+1}(x_1, \dots, x_k) + \cdots + c_N\, e_N(x_1, \dots, x_k). \end{aligned}$$

By the above local argument, the set of $b \in \mathbb{R}^k$ such that $(f^c)_a$ is not Morse, has measure zero. As a union of countably many such sets, the set of $(b_1, \ldots, b_k, c_{k+1}, \ldots, c_N) \in \mathbb{R}^N$ such that $(f^c)_b$ is not Morse, also has measure zero.

Thus, for each each α there is a measure zero set of $a \in \mathbb{R}^N$ so that f_a is not Morse on U_α. Since a countable union of measure zero subset still has measure zero, there is a dense set of $a \in \mathbb{R}^N$ so that f_a is Morse on all of M. $\square$

29.2 The first fundamental theorem of Morse theory

Let M be a manifold without boundary and $f\colon M \to \mathbb{R}$ be a Morse function. We shall study M by studying the (sub)level sets

$$M_{\leqslant a} := f^{-1}((-\infty, a]) \qquad \text{and} \qquad M_a := f^{-1}(\{a\}).$$

By the submersion theorem, if a is a regular value, $M_{\leqslant a} \subset M$ is a codimension zero submanifold with boundary $\partial M_{\leqslant a} = M_a$ given by a level set.

29.2.1 Gradients

Given a smooth function $f\colon \mathbb{R}^k \to \mathbb{R}$, its gradient is the vector field

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_k} \end{bmatrix}.$$

That is, the component in the direction of the standard basis vector e_i is given by $\frac{\partial f}{\partial x_i}$. Using the standard Riemannian metric, we can identify each basis vector of $\mathbb{R}^k$ with a basis vector of its dual $(\mathbb{R}^k)^*$: e_i corresponds to the linear functional $\langle e_i, - \rangle$. In other words, the Riemannian metric provides an isomorphism of the tangent spaces to points in $\mathbb{R}^k$ with the corresponding cotangent spaces. From a vector field, a section of the tangent bundle, we thus get a 1-form, a section of the cotangent bundle. In this particular case, the Riemannian metric sends e_i to dx_i, and we see that ∇f gets sent to

$$df = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i} dx_i.$$

This discussion extends to manifolds with a Riemannian metric g. This Riemannian metric is given by a smoothly varying non-degenerate bilinear form on the tangent space $T_p(M)$,

$$T_p(M) \times T_p(M) \ni (v, w) \longmapsto g(v, w) \in \mathbb{R}$$

and thus provides an isomorphism of vector bundles $TM \to T^*M$

$$T_p(M) \ni v \longmapsto g(v, -) \in T_p^*(M).$$

In particular, it sends sections of TM to sections of T^*M and vice versa: every vector field corresponds to a unique 1-form.

Now suppose we have a smooth function $f\colon M \to \mathbb{R}$, then there is a 1-form $df \in \Omega^1(M)$. The Riemannian metric sends this to a vector field ∇f, which we call *the gradient of* f (this notation and terminology is not ideal, as the gradient depends on the choice of Riemannian metric).

29.2.2 Gradient flow without critical points

Suppose that M is compact, then using the techniques of the previous lecture, we can flow along ∇f. The result is a smooth family of diffeomorphisms $\phi_t\colon M \to M$ for $t \in \mathbb{R}$, satisfying $\phi_0 = \mathrm{id}$, $\phi_s \circ \phi_t = \phi_{s+t}$ and $\frac{d}{dt}\phi_t = \nabla f$.

To understand this flow, let us see how f varies over an integral curve $\phi_t(p)$. Let $||-||^2$ denote the norm on T^*M coming from the Riemannian metric, then we compute that

$$\begin{aligned}\frac{d}{dt}f(\phi_t(p))|_{t=0} &= d_pf(\frac{\partial \phi_t(p)}{\partial t}|_{t=0})\\ &= d_pf(\nabla f(p))\\ &= ||d_pf||^2.\end{aligned}$$

Since ϕ_t is a flow, this implies that $\frac{d}{dt}f(\phi_t(p))|_{t=s} = ||d_{\phi_s}(p)f||^2$. We conclude that:

Lemma 29.2.1. *The function $t \mapsto f(\phi_t(p))$ is non-decreasing and strictly increasing when $\phi_t(p)$ is not a critical point.*

We shall use this to study the subset

$$M_{[a,b]} := f^{-1}([a,b]),$$

for $a < b$ regular values. This is a codimension zero submanifold of M with boundary $M_a \sqcup M_b$. Let us take $p \in M_{[a,b]}$ and consider the integral curve $\phi_t(p)$. When does this leave $M_{[a,b]}$?

Lemma 29.2.2. *Fix $p \in M_a$. Let $(0,c)$ for $c > 0$ be the maximal interval such that $\phi_t(p) \in \mathrm{int}(M_{[a,b]})$ for $t \in (0,c)$. Then if c is finite, $\phi_c(p) \in M_b$, and if $c = \infty$ then there are $t_i \to \infty$ such that $\phi_{t_i}(p)$ converges to a critical point.*

Proof. Suppose that c is finite. Then we know that $\phi_c(p)$ is defined but not in $\mathrm{int}(M_{[a,b]})$ (or we could extend the interval $(0,c)$). Thus it is either in M_a or M_b, and since a is not a critical value, $f(\phi_t(p))$ is strictly increasing with t at $t = 0$. It is non-decreasing afterwards, so we must have that $\phi_c(p) \in M_b$.

If $c = \infty$, then since $f(\phi_t(p))$ increases at $t \to \infty$ but remains strictly smaller than b,

$$\int_0^N ||d_{\phi_t(p)}f||^2 dt = \int_0^N \frac{d}{dt}f(\phi_t(p))dt = f(\phi_N(p)) - f(\phi_0(p))$$

converges as $N \to \infty$. Thus $||d_{\phi_t(p)}f||$ must decrease to 0 as t increases. This means that it eventually be contained in any open neighbourhood of the critical points in $M_{[a,b]}$. Since the $M_{[a,b]}$ is compact, this means that we can find a subsequence which converges to a critical point. □

We are interested in the case that there is no critical point in $M_{[a,b]}$, and thus the second case in the above lemma can not occur. The same argument then tells us that when we start any $p \in M_{[a,b]}$, there is some maximal finite interval (c', c) with $c' < c$ such that $\phi_t(p) \in \mathrm{int}(M_{[a,b]})$ for $t \in (c', c)$ and $\phi_{c'}(p) \in M_a$ and $\phi_c(p) \in M_b$.

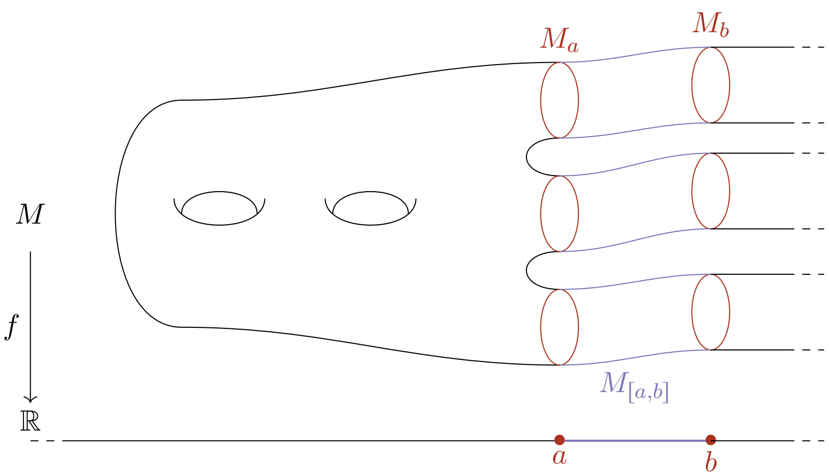


Figure 29.1 An example of a proper map $f\colon M \to \mathbb{R}$ such that $M_{[a,b]}$ contains no critical point. note that $M_{(-\infty,a]}$ contains 7 critical points.

Theorem 29.2.3 (First fundamental theorem of Morse theory)**.** *If the interval $[a, b]$ contains no critical point, then there is a diffeomorphism $M_{[a,b]} \to M_a \times [0, 1]$ which restricts to the map $M_a \to M_a \times \{0\}$ given by $p \mapsto (p, 0)$.*

Proof. By the previous lemma, for each $p \in M_a$ there is a $c(p) > 0$ such that $\phi_{c(p)}(p) \in M_b$. This is unique because $f(\phi_t(p))$ is non-decreasing and strictly increases at $t = c(p)$. By smooth dependence of solutions of ordinary differential equations on initial conditions, $c\colon M_a \to (0, \infty)$ is smooth. Now consider the map

$$\begin{aligned} \Psi\colon M_a \times [0,1] &\longrightarrow M_{[a,b]} \\ (p,t) &\longmapsto \phi_{tc(p)}(p). \end{aligned}$$

In other words, it is the composition of the diffeomorphism $(p, t) \mapsto (p, tc(p))$ between $M_a \times [0, 1]$ and $N := \{(p, t) \in M_a \times \mathbb{R} \mid 0 \leqslant t \leqslant c(p)\}$ and the smooth map $\phi\colon M_a \times \mathbb{R} \to M$.

It has an inverse given as follows: given by $p \in M_{[a,b]}$ take (c', c) as above and define $\Phi(p) := (\phi_{c'}(p), -c')$. This is smooth using the smooth dependence of solutions of ordinary differential equations on initial conditions and smoothness of ϕ. It is an inverse by uniqueness of solutions to ordinary differential equations. $\square$

Corollary 29.2.4. *If the interval $[a, b]$ contains no critical points, then $M_{\leqslant a}$ is diffeomorphic to $M_{\leqslant b}$.*

Proof. $M_{\leqslant b}$ is obtained from $M_{\leqslant a}$ by gluing on $M_{[a,b]}$. Recall that the existence of collars tells us that $M_{\leqslant a}$ contains a neighborhood C of M_a with a diffeomorphism $M_a \times [-1,0] \to C$. Since $M_{[a,b]} \cong M_a \times [0,1]$ by the previous theorem, we see that $M_{\leqslant b}$ is diffeomorphic to $M_{\leqslant a}$ via

$$\begin{aligned} M_{\leqslant b} &\longrightarrow M_{\leqslant a} \\ p &\longmapsto \begin{cases} c(q,\eta(t)) & \text{if } p = c(q,t) \in C, \\ c(q,\eta(t)) & \text{if } p = \Psi(q,t) \in M_{[a,b]}, \\ p & \text{if } p \in M_{\leqslant a} \backslash C. \end{cases} \end{aligned}$$

with $\eta\colon [-1,1] \to [-1,0]$ a diffeomorphism which is the identity near -1. $\square$

Chapter 30

Second fundamental theorem of Morse theory

In this lecture we discuss the part of Morse theory which does involves critical points and show if $[a, b] \subset \mathbb{R}$ contains a single critical points of f of index p, then $f^{-1}((-\infty, b])$ is obtained by attaching an i-handle to $f^{-1}((-\infty, a])$. This can be found in Section 5.1 of [Wal16] and Chapter I.§3 of [Mil63].

Remark 30.0.1. Throughout this chapter we shall ignore the issue of "smoothing corners." If you want to understand these technical details, see Section 2.6 of [Wal16].

30.1 The second fundamental theorem of Morse theory

Let M be a compact manifold and $f\colon M \to \mathbb{R}$ be a Morse theory. We recall some notation from the previous lecture

$$M_a := f^{-1}(\{a\}), \qquad M_{\leqslant a} := f^{-1}((-\infty, a]) \qquad \text{and} \qquad M_{[a,b]} := f^{-1}([a, b]).$$

In the previous chapter we saw that if there is no critical value in $[a, b]$—or equivalently no critical point in $M_{[a,b]}$—then there is a diffeomorphism $M_{[a,b]} \to M_a \times [a, b]$ that is the identity on M_a.

30.1.1 The Morse lemma

What happens when there is a unique non-degenerate critical point p in $M_{[a,b]}$? Pick a chart $\phi\colon \mathbb{R}^k \supset U \to V \subset M$ such that $\phi(0) = p$, and in terms of coordinates $(x_1, \ldots, x_k) \in U$, f is given by

$$f(x_1, \ldots, x_k) = c - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{k} x_i^2.$$

This is possible by the Morse lemma, and we provide a proof below from [DK04a, Theorem 4.8.1] that is different from the one in [GP10]:

Lemma 30.1.1. *If a critical point $p \in M$ of $f\colon M \to \mathbb{R}$ is non-degenerate then there exists a chart as above.*

Proof. Without loss of generality we may assume that $f(p) = 0$, and fix a chart $\phi\colon \mathbb{R}^k \supset U \to V \subset M$ such that $\phi(0) = p$. Let $x = (x_1, \ldots, x_k)$ denote the coordinates near p coming from this chart, defined on $U \subset \mathbb{R}^k$.

Let $\mathrm{Sym}(\mathbb{R}^k)$ denote the space of symmetric $(k \times k)$-matrices over $\mathbb{R}$; we give this a smooth structure by using the entries to identify with a Euclidean space. The multi-variable version of Taylor approximation says that there is a smooth map $Q\colon W \to \mathrm{Sym}(\mathbb{R}^k)$ such that $f(x) = \langle Q(x)x, x\rangle$ and which satisfies $Q(0) = \mathrm{Hess}_0(f)$ [DK04a, Theorem 2.8.3]. We first want to change coordinates from x to y so that Q is independent of y. To do this, make the ansatz that $y = A(x)x$ for a smooth map $A\colon U \to \mathrm{GL}_k(\mathbb{R})$. In that case we need to solve the equation

$$\langle Q(0)A(x)x, A(x)x\rangle = \langle Q(x)x, x\rangle,$$

or equivalently $A^t(x)Q(0)A(x) = Q(x)$. We then consider the smooth map $G\colon \mathrm{Sym}(\mathbb{R}^k) \times U \to \mathrm{Sym}(\mathbb{R}^k)$ given by

$$(B, x) \mapsto \left(\mathrm{id} + \frac{1}{2}Q(0)^{-1}B\right)^t Q(0)\left(\mathrm{id} + \frac{1}{2}Q(0)^{-1}B\right) - Q(x).$$

This is equal to 0 at $(B, x) = (0, 0)$ and its derivative with respect to B at $(B, x) = (0, 0)$ is the identity:

$$\begin{aligned}\frac{\partial}{\partial B}G(B, 0) &= \left(\frac{1}{2}Q(0)^{-1}\right)^t Q(0) + Q(0)\left(\frac{1}{2}Q(0)^{-1}\right) \\ &= \frac{1}{2}\mathrm{id} + \frac{1}{2}\mathrm{id} = \mathrm{id}.\end{aligned}$$

By the implicit function theorem, there exists a neighbourhood U' of 0 in U and a smooth map $\beta\colon U \to \mathrm{Sym}(\mathbb{R}^k)$ such that $G(\beta(x), x) = 0$. Taking

$$A(x) \coloneqq \mathrm{id} + \frac{1}{2}Q(0)^{-1}\beta(x)$$

we obtain that $\langle Q(0)A(x)x, A(x)x\rangle = \langle Q(x)x, x\rangle$. So we shall use coordinates $y = A(x)x$. Since $x \mapsto A(x)x$ has derivative id at 0, by the inverse function theorem there exists some smaller neighbourhood U'' on which this map is a diffeomorphism.

Now that in y-coordinates we have that $f(y) = \langle Q(0)y, y\rangle$, it is a matter finding a matrix A such that $A^tQ(0)A$ diagonal with entries ± 1 and using the coordinates $z = Ay$ instead. This is possible by Gram-Schmidt. □

Remark 30.1.2. The proof in fact tells us we can take $A(x)$ to be an invertible symmetric matrix.

30.1.2 The second fundamental theorem

Let $\epsilon > 0$ be small enough such that U contains the ball $B_{\sqrt{2\epsilon}}(0)$ and $a < c - 2\epsilon < c + 2\epsilon < b$. Then we shall describe the difference between $f^{-1}([a, c - \epsilon])$ and $f^{-1}([a, c + \epsilon])$, at first up to homotopy and then as a manifold. To do so, define the subset $C \subset B_{\sqrt{2\epsilon}}(0)$ by $\{(x_1, \ldots, x_\lambda, 0, \ldots, 0) \mid \sum_{i=1}^{\lambda} x_i^2 \leqslant \epsilon\}$, where C stands for *core*. This is of course a λ-dimensional disk, whose boundary $(\lambda-1)$-sphere lies in $f^{-1}(c-\epsilon)$. The description of $M_{[a,c+\epsilon]}$ up to homotopy equivalence is as follows, and along the way we will in fact obtain a description up to diffeomorphism. This amounts to two applications of the first fundamental theorem of Morse theory, combined with a difficult computation in the local model provided by the Morse lemma.

Proposition 30.1.3. *$M_{[a,c+\epsilon]}$ is homotopy equivalent, as a topological space, to the union $M_{[a,c-\epsilon]} \cup C$.*

We shall use the notion of a deformation retraction: for $A \subset X$ closed, a deformation retraction of X onto A is a homotopy $H\colon X \times [0,1] \to X$ such that $H(x,1) \in A$ for all $x \in X$ and $H(a,t) = a$ for all $a \in A$ and $t \in [0,1]$. If there is a deformation retraction of X onto A, then $i\colon A \hookrightarrow X$ is a homotopy equivalence; its homotopy inverse is $H(-,1)$.

To prove the proposition, we shall find a neighbourhood U of $M_{[a,c-\epsilon]} \cup C$ which is a deformation retract $M_{[a,c+\epsilon]}$ and itself deformation retracts onto $M_{[a,c-\epsilon]} \cup C$:

$$M_{[a,c-\epsilon]} \cup C \overset{\simeq}{\hookrightarrow} U \overset{\simeq}{\hookrightarrow} M_{[a,c+\epsilon]}.$$

To do so, we modify f to another function F with some special properties. We will only change f on the subset $M_{[c-\epsilon,c+\epsilon]}$, using a smooth function $\phi\colon [0,\infty) \to [0,\infty)$ satisfying

(i) $\phi(0) \in (\epsilon, 2\epsilon)$,

(ii) $\phi(t) = \phi(0)$ for t near 0,

(iii) $\phi(t) = 0$ for $t \in [2\epsilon, \infty)$, and

(iv) $\phi'(t) \in (-1, 0]$ for all $t \in [0,\infty)$.

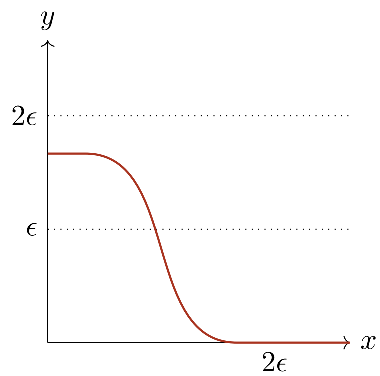


Figure 30.1 The function ϕ.

Then the function F shall be given by

$$F\colon M \to \mathbb{R}$$
$$x \mapsto \begin{cases} f(x) - \phi\left(\sum_{i=1}^{\lambda} x_i^2 + 2\sum_{i=\lambda+1}^{k} x_i^2\right) & \text{if } x \in V, \\ f(x) & \text{otherwise.} \end{cases}$$

This is a smooth function because $\phi\left(\sum_{i=1}^{\lambda} x_i^2 + 2\sum_{i=\lambda+1}^{k} x_i^2\right)$ has compact support in V. It is essentially f with a strip near the critical point pushed downwards:

Lemma 30.1.4. *F has the following properties:*

(1) $M_{[a,c+\epsilon]} = F^{-1}([a, c+\epsilon])$.

(2) F has the same critical points as f.

(3) In $B_{\sqrt{2\epsilon}}(0) \subset U$, $F^{-1}([a, c-\epsilon])$ is described by Figure 30.2. More precisely, U is diffeomorphic to $M_{[a,c-\epsilon]} \cup (D^\lambda \times D^{k-\lambda})$ attached along an embedding $\partial D^\lambda \times D^{k-\lambda}$ (up to smoothing corners), with C corresponding to $D^\lambda \times \{0\}$.

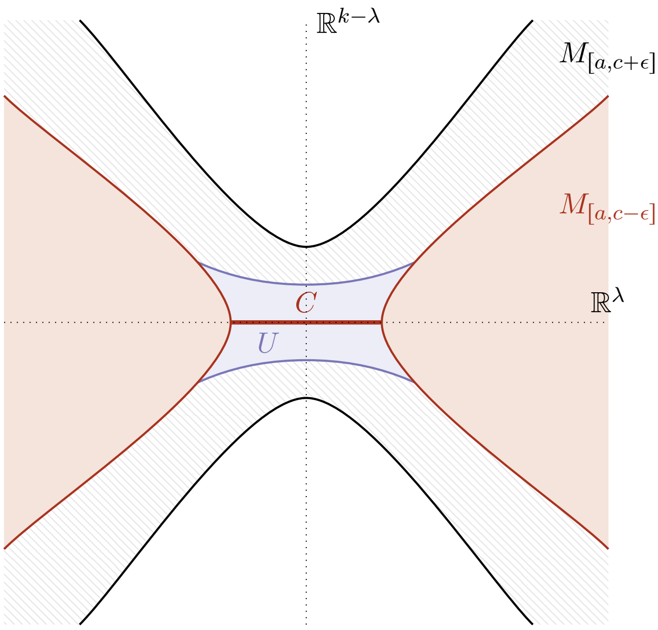


Figure 30.2 The set U is the union of the red and purple parts. The set is $f^{-1}([a, c+\epsilon])$ is the union of the red, purple and dashed parts.

Proof. Let us write $x = (y, z)$ when $x \in U$, with $y = (y_1, \ldots, y_\lambda)$ denoting the first λ coordinates and $z = (z_1, \ldots, z_{k-\lambda})$ denoting the remaining $k - \lambda$.

Part (1) follows by noting that since $F \leqslant f$ (since ϕ is non-negative), we have that $f^{-1}([a, c+\epsilon]) \subset F^{-1}([a, c+\epsilon])$. For the converse, if $x \in F^{-1}([a, c+\epsilon])$ and

$\phi(||y||^2 + 2||z||^2) > 0$, then $||y||^2 + 2||z||^2 < 2\epsilon$ (since $\phi(t) = 0$ when $t \geqslant 2\epsilon$), so that

$$f(x) - f(c) = -||y||^2 + ||z||^2 \leqslant \frac{1}{2}||y||^2 + ||z||^2 < \epsilon$$

and thus $x \in f^{-1}([a, c+\epsilon])$ as well.

For part (2) there is only something to check when $p \in V$. Working in local coordinates, we have that $\frac{1}{2}\nabla F(x) = (-y - \phi'(x)y, z - \phi'(x)2z)$. This certainly vanishes at 0, so p is a critical point. To see this is the only critical point, note that since $\phi'(x) > -1$, we must have $y = 0$ and since $\phi'(x) \leqslant 0$, we must have $z = 0$.

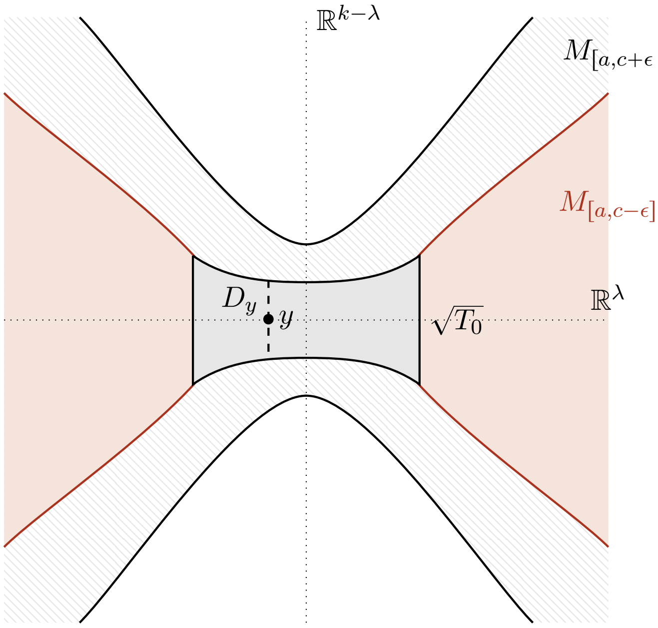


Figure 30.3 The gray part consists of those disks D_y in the proof of Lemma 30.1.4 that do not coincide with those for the original function f.

The precise proof of part (3) is a rather long computation, as we need to produce an explicit diffeomorphism; details can be found in Chapter 3 of [Mil63] or Section VII.2.2 of [Kos93]. The main observation is that upon fixing the first λ-coordinates to be equal to $y = (y_1, \ldots, y_\lambda)$ with $||y||^2 \leqslant \epsilon$, the intersection of $F^{-1}([a, c-\epsilon])$ with the $(k-\lambda)$-dimensional plane $\{y\} \times \mathbb{R}^{k-\lambda}$ is given by a disk whose radius depends smoothly on y. Of course, as soon as $||y||^2 + 2||z||^2$ reaches $T_0 := \inf\{t \mid \phi(t) = 0\}$, then this disk coincides with the intersection of the original set $f^{-1}([a, c-\epsilon])$ with the $(k-\lambda)$-dimensional plane $\{y\} \times \mathbb{R}^{k-\lambda}$.

To check this, note that this intersection is given by the set $(y, z) \in \mathbb{R}^\lambda \times \mathbb{R}^{k-\lambda}$ with z satisfying

$$c - ||y||^2 + ||z||^2 - \phi(||y||^2 + 2||z||^2) \leqslant c - \epsilon.$$

The condition may be rewritten in terms of $\alpha(y,z) := ||y||^2 + 2||z||^2$ as

$$\phi(\alpha(y,z)) - \alpha(y,z)/2 \geqslant \epsilon - \frac{3}{2}||y||^2. \tag{30.1}$$

Since $\phi(t) - t/2$ is decreasing on the interval $[0, 2\epsilon]$ from $\phi(0) > \epsilon$ to $-\epsilon$, there is a unique $t_0 > 0$ such that $\phi(t_0) - t_0/2 = \epsilon - \frac{3}{2}||y||^2$. In terms of t_0, the inequality (30.1) is equivalent to

$$||z||^2 \leqslant \frac{1}{2}(t_0 - ||y||^2). \tag{30.2}$$

Since $\phi(0) > \epsilon$ and $\phi'(t) > -1$, we have that $\phi(t_0) > \epsilon - t_0$, so that we have $\phi(t_0) - t_0/2 > \epsilon - \frac{3}{2}t_0$ and thus that $t_0 > ||y||^2$, so the right hand side of (30.2) is strictly positive. The set $D_y := \{(y,z) \mid ||z||^2 \leqslant \frac{1}{2}(t_0 - ||y||^2)\}$ is the desired disk. □

We shall then define $U = F^{-1}([a, c-\epsilon])$, which is diffeomorphic to $M_{[a,c+\epsilon]}$. To see this, apply the first fundamental theorem of Morse theorem using the observation that there is no critical point in $M_{[a,c+\epsilon]} \backslash U$. From this observation and part (3) of the Lemma, we not only obtain the homotopy-theoretic description also the stronger statement that $M_{[a,c+\epsilon]}$ is diffeomorphic to $(M_a \times [a, c-\epsilon]) \cup (D^\lambda \times D^{k-\lambda})$. Thus we have proven:

Theorem 30.1.5 (Second fundamental theorem of Morse theory)**.** *If $M_{[a,b]}$ contains a unique non-degenerate critical point in its interior, which has index λ, then there is a diffeomorphism (up to smoothing corners)*

$$M_{(-\infty,b]} \xrightarrow{\cong} M_{(-\infty,a]} \cup_{\partial D^\lambda \times D^{k-\lambda}} (D^\lambda \times D^{k-\lambda}).$$

30.1.3 Handle decompositions

The construction which takes a manifold W with boundary ∂W and an embedding $e\colon \partial D^\lambda \times D^{k-\lambda} \hookrightarrow \partial W$ to the manifold obtained by smoothing the corners in

$$W \cup_{\partial D^\lambda \times D^{k-\lambda}} D^\lambda \times D^{k-\lambda},$$

is called a *handle attachment of index λ.*

The second fundamental theorem of Morse theory says that each critical point of index λ corresponds to a handle attachment of index λ, as long as all critical points have distinct critical values. This is a minor restriction, as by a small perturbation we may assume this is the case, cf. Exercise 1.§7.19 of [GP10].

Since every manifold admits a Morse function and Morse singularities are isolated, we conclude that every compact manifold M can be obtained by a finite number of handle attachments. We say it admits a *handle decomposition.*

Example 30.1.6. The height function

$$\begin{aligned} S^k &\longrightarrow \mathbb{R} \\ (x_0, \ldots, x_k) &\longmapsto x_0 \end{aligned}$$

is a Morse function with a minimum at $(-1, 0, \ldots, 0)$ (so index 0) and a maximum at $(1, 0, \ldots, 0)$ (so index k). Thus we see that S^k has a handle decomposition with a single 0- and k-handle. This is just the decomposition

$$S^k = (D^0 \times D^k) \cup_{\partial D^k \times D^0} (D^k \times D^0)$$

into two hemispheres.

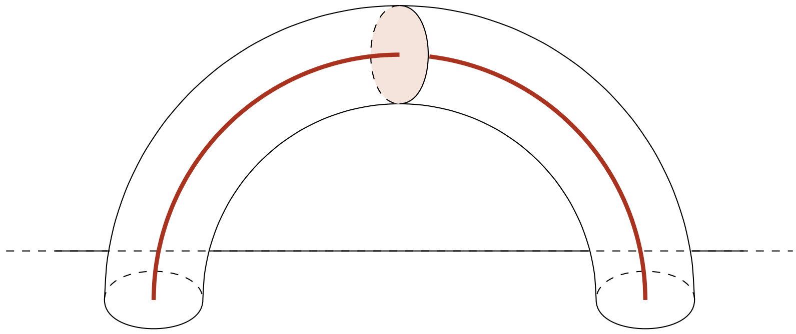


Figure 30.4 A 3-dimensional 1-handle $D^1 \times D^2$ attached to $\mathbb{R}^2 = \partial(\mathbb{R}^2 \times (-\infty, 0])$. The red line is $D^1 \times \{0\}$, the orange disk is $\{0\} \times D^2$.

30.2 Morse functions and de Rham cohomology

The relationship between de Rham cohomology and Morse functions will be the following:

Proposition 30.2.1. *Let $f\colon M \to \mathbb{R}$ be a Morse function on a k-dimensional compact manifold M, then for each $0 \leqslant \lambda \leqslant k$ there is an inequality*

$$\#\{\textit{critical points of } f \textit{ of index } \lambda\} \geqslant \dim H^\lambda(M).$$

Proof. We may assume without loss of generality that f has critical points with distinct critical values, and shall ignore the smoothing of corners in this proof. Pick $a_0 < \cdots < a_n$ such that $f(M) \subset [a_0, a_n]$ and each interval $[a_{i-1}, a_i]$ contains a unique critical value.

We shall prove by induction over i that there is an inequality

$$\#\{\text{critical points of } f|_{M_{(-\infty,a_i]}} \text{ of index } \lambda\} \geqslant \dim H^\lambda(M_{(-\infty,a_i]}).$$

The initial case is $i = 0$, and then $M_{(-\infty,a_0]} = \varnothing$ and the statement is clearly true. For the induction step, we use the second fundamental theorem of Morse theory:

$$M_{(-\infty,a_i]} \cong M_{(-\infty,a_{i-1}]} \cup_{\partial D^\lambda \times D^{k-\lambda}} D^\lambda \times D^{k-\lambda}.$$

Let us apply Mayer–Vietoris to the open cover

$$U = \operatorname{int}(D^\lambda) \times D^{k-\lambda},$$

$$V = M_{(-\infty,a_{i-1}]} \cup_{\partial D^\lambda \times D^{k-\lambda}} (D^\lambda \backslash D^\lambda_{1/2}) \times D^{k-\lambda}.$$

Then U is contractible, V is homotopy equivalent to $M_{(-\infty,a_{i-1}]}$ and $U \cap V$ is homotopy equivalent to $S^{\lambda-1}$.

From the Mayer–Vietoris long exact sequence we conclude that

$$H^i(M_{(-\infty,a_i]}) \longrightarrow H^i(M_{(-\infty,a_{i-1}]})$$

is an isomorphism unless $i = \lambda, \lambda - 1$. In those cases, we get an exact sequence (for convenience we assume $\lambda \geqslant 3$, dealing with H^0's requires a bit of additional care)

$$\begin{tikzcd}
H^\lambda(M_{(-\infty,a_i]}) \arrow[r] & H^\lambda(M_{(-\infty,a_{i-1}]}) \arrow[r] & 0 \\
H^{\lambda-1}(M_{(-\infty,a_i]}) \arrow[r] & H^{\lambda-1}(M_{(-\infty,a_{i-1}]}) \arrow[r] & \mathbb{R} \arrow[ull, rounded corners] \\
& & 0 \arrow[ull, rounded corners]
\end{tikzcd}$$

Two things can happen to the $\mathbb{R}$ in $H^{\lambda-1}(U \cap V)$: either it adds to H^λ

$$\begin{aligned}
\dim H^\lambda(M_{(-\infty,a_i]}) &= \dim H^\lambda(M_{(-\infty,a_{i-1}]}) + 1, \\
\text{and } \dim H^{\lambda-1}(M_{(-\infty,a_i]}) &= \dim H^{\lambda-1}(M_{(-\infty,a_{i-1}]}),
\end{aligned}$$

or it subtracts from $H^{\lambda-1}$,

$$\begin{aligned}
\dim H^\lambda(M_{(-\infty,a_i]}) &= \dim H^\lambda(M_{(-\infty,a_{i-1}]}), \\
\text{and } \dim H^{\lambda-1}(M_{(-\infty,a_i]}) &= \dim H^{\lambda-1}(M_{(-\infty,a_{i-1}]}) - 1.
\end{aligned}$$

In both cases the inequalities to be proven are satisfied. (Indeed, it may be helpful to observe that equality occurs only if all critical points add cohomology and never subtract cohomology). ☐

Example 30.2.2. We know the cohomology of the 2-torus: $H^0(\mathbb{T}^2) = \mathbb{R}$, $H^1(\mathbb{T}^2) = \mathbb{R}^2$, $H^2(\mathbb{T}^2) = \mathbb{R}$. Thus every Morse function on $\mathbb{T}^2$ has at least one minimum, one maximum, and two saddle points. We leave it to you to find an example of such a Morse function.

Example 30.2.3. It is not true that you can always find a Morse function with exactly $\dim H^\lambda$ critical points of index λ. For example, only $H^0(\mathbb{R}P^2) = \mathbb{R}$ is non-zero, but since $\mathbb{R}P^2$ is compact every Morse function on it has a maximum.

Remark 30.2.4. Given a Morse function $f\colon M \to \mathbb{R}$, there is a chain complex C^f_* with C^f_p given by the free $\mathbb{R}$-vector space on the critical points of f of index p, and differential given by counting flowlines. Its homology is the *Morse homology* $H_*(M; f)$. It turns out to be independent of f and for compact M there is an isomorphism $H_p(M; f)^* \cong H^p(M)$.

Chapter 31

Classification of smooth surfaces

In this lecture we use the theory of handle decompositions to classify smooth compact surfaces.

31.1 Manipulating handle decompositions

31.1.1 Handle decompositions

We recall from the previous lecture the notion of a handle attachment. The input is a smooth manifold W with boundary ∂W and an embedding $\phi\colon \partial D^\lambda \times D^{k-\lambda} \hookrightarrow \partial W$ that we call the *attaching map*, and the output is the smooth manifold

$$W + (\phi) := W \cup_{\partial D^\lambda \times D^{k-\lambda}} D^\lambda \times D^{k-\lambda},$$

where the identification is made along ϕ and implicitly smooth the corners. We name some subspaces of $W + (\phi)$:

- $D^\lambda \times D^{k-\lambda}$ is the *handle*,
- $D^\lambda \times \{0\}$ is the *core*,
- $\partial D^\lambda \times \{0\}$ is the *attaching sphere*,
- $\{0\} \times D^{k-\lambda}$ is the *cocore*,
- $\{0\} \times \partial D^{k-\lambda}$ is the transverse sphere.

To give a handle decomposition of a manifold is write it as diffeomorphic to one obtained by iterating handle attachments. The existence of Morse functions implies that every compact manifold admits a finite handle decomposition, so we can always write

$$M \cong (\phi_0) + \cdots + (\phi_r)$$

for some r. Note that this notation is a bit deceptive, since handle attachments do not commute.

Remark 31.1.1. It is convenient to observe that handle decompositions can be read backwards: we think of a λ-handle $D^\lambda \times D^{k-\lambda}$ rather as a $(k-\lambda)$-handle and reverse the order of handle attachments. This amounts to replacing a Morse function $f\colon M \to \mathbb{R}$ with its negative $-f\colon M \to \mathbb{R}$.

31.1.2 Handle manipulations

There are four moves to modify handle decompositions:

(1) Handle isotopy.

(2) Handle rearrangement.

(3) Handle cancellation and addition.

(4) Handle exchange.

We will need only the first three of these.

Handle isotopy

The first concerns modify the attaching map:

Lemma 31.1.2. *If ϕ is isotopic to ϕ' then there is a diffeomorphism*

$$W + (\phi) \cong W + (\phi').$$

Proof. Let $\phi_t \colon D^\lambda \times D^{k-\lambda} \hookrightarrow \partial W$ be an isotopy of embeddings from $\phi_0 = \phi$ to $\phi_1 = \phi'$, and use the isotopy extension theorem to find an isotopy of diffeomorphisms $f_t \colon \partial W \to \partial W$ so that (i) $f_0 = \mathrm{id}_{\partial W}$ and (ii) $f_t\phi_0 = \phi_t$. Picking a closed collar $\chi \colon \partial W \times [0,1] \hookrightarrow W$ we then define a diffeomorphism

$$\begin{aligned} F \colon W + (\phi) &\longrightarrow W + (\phi') \\ p &\longmapsto \begin{cases} (f_{1-t}(q), t) & \text{if } p = \chi(q,t) \\ p & \text{else} \end{cases} \end{aligned}$$

which is well-defined since it sends a point in the image of ϕ to the corresponding point in the image of ϕ'. □

Handle rearrangement

The second concerns the ordering the handles. We start with the following obvious observation:

Lemma 31.1.3. *If ϕ_0 and ϕ_1 have disjoint image in ∂W, then there is a diffeomorphism*

$$W + (\phi_0) + (\phi_1) \cong W + (\phi_1) + (\phi_0).$$

It is sometimes possible to arrange the hypothesis:

Lemma 31.1.4. *If* $\mathrm{index}(\phi_0) \geqslant \mathrm{index}(\phi_1)$ *then we can isotope ϕ_1 to ϕ_1' which takes image in $\partial W \backslash \mathrm{im}(\phi_0)$ and thus*

$$W + (\phi_0) + (\phi_1) \cong W + (\phi_1') + (\phi_0).$$

Proof. Write $\lambda_0 := \mathrm{index}(\phi_0)$ and $\lambda_1 := \mathrm{index}(\phi_1)$. There are three steps:

1. We first isotope the attaching sphere of ϕ_1 to be disjoint from the transverse sphere of ϕ_0. To do so, we make the former transverse to the latter by an isotopy. Since the first is a (λ_1-1)-sphere, the latter is a $(k-\lambda_0-1)$-sphere, and we are in the $(k-1)$-dimensional manifold $\partial(W+(\phi_0))$, they are in fact disjoint.
2. We next isotope the attaching sphere of ϕ_1 to be disjoint from ϕ_0. To do so, we choose a vector field on $\partial(W+(\phi_0))\backslash(\{0\}\times D^{k-\lambda_0})$ that on $(D^{\lambda_0}\times\partial D^{k-\lambda_0})\backslash(\{0\}\times\partial D^{k-\lambda_0})$ is given by $\partial/\partial r$ in the first coordinate. Flowing along at least one unit of time will move the compact submanifold $\phi_1(\partial D^{\lambda_1}\times\{0\})$ out of $D^{\lambda_0}\times\partial D^{k-\lambda_0}\subset\partial(W+(\phi_0))$.
3. We finally shrink the $D^{k-\lambda_1}$-direction so that ϕ_1 is disjoint from ϕ_0. □

Applying this inductively, we can find for every compact manifold M a handle decomposition

$$M\cong(\phi_0)+\cdots+(\phi_r)$$

where $\mathrm{index}(\phi_{i-1})\leqslant\mathrm{index}(\phi_i)$ and we can assume all handles of the same index are attached simultaneously.

Handle cancellation

The third concerns the removal of a pair of handles. We will not give a proof since we will only need it in a special case:

Proposition 31.1.5. *If the attaching sphere of ϕ_1 intersects the transverse sphere of ϕ_0 transversally in a single point, then there is a diffeomorphism*

$$W+(\phi_0)+(\phi_1)\cong W.$$

The special case we need is that of 2-dimensional manifolds. First consider the case that ϕ_0 is a 0-handle, i.e. a disc D^2, and ϕ_1 is a 1-handle. The transverse sphere of $\phi_0\cong D^2$ is all of ∂D^2 and the attaching sphere of $\phi_1\cong D^1\times D^1$ is ∂D^1 consisting of two points. So the hypothesis is simply that the strip $D^1\times D^1$ is attached along a single line segment to D^2, and the other line segment is attached to W. We are thus just gluing a D^2 to W along half of its boundary and smoothing corners, and this is diffeomorphic to W again. For the last step we use the following lemma to reduce it to writing down a diffeomorphism in a local model, by choosing e to be the unit disc in a chart:

Lemma 31.1.6. *Let M be a connected d-dimensional manifold with an embedding $e\colon D^d\to M$. Let $\phi\colon D^d\to M$ ben another embedding and suppose that either (i) M is oriented and both e and ϕ are orientation-preserving, (ii) M is non-orientable.*

Proof. We prove the first case, as the second is similar. Applying isotopy extension to a path connecting $\phi(0)$ to $e(0)$ to isotope ϕ and shrinking its domain, we can isotope ϕ to a ϕ' with image in $\mathrm{im}(e)$. Thus we may assume that $M=D^d$ and we have proved before that every orientation-preserving embedding $\phi\colon D^d\to D^d$ is isotopic to the identity. □

The second case that ϕ_1 is a 1-handle and 2-handle is similar.

31.2 The classification of surfaces

We will use the existence of handle decompositions and the above three moves to classify compact surfaces.

31.2.1 The orientable case

You should be familiar with the 2-sphere and the 2-torus, which are the case $g = 0$ and $g = 1$ of the surface Σ_g of genus g. The latter admits a handle decomposition

$$\Sigma_g = (\phi_0) + \sum_{i=1}^{2g} (\phi_i) + (\phi_{2g+1})$$

where (ϕ_0) has index 0, (ϕ_i) has index 1, and ϕ_{2g+1} has index 2, and the attaching maps for the 1-handles alternate in the following pattern:

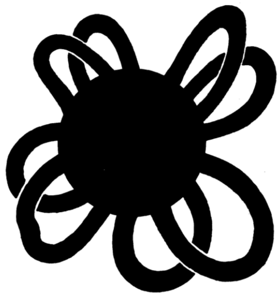


Figure 31.1 An attachment pattern for a genus 4-surface (from [FM97]).

It is in fact unnecessary to describe the attaching map of the 2-handle, using the following lemma and the fact that there is a diffeomorphism of D^2 that restricts to an orientation-reversing diffeomorphism of its boundary (namely, reflection):

Lemma 31.2.1. *Every orientation-preserving diffeomorphism of S^1 is isotopic to the identity.*

Proof. Let $f\colon S^1 \to S^1$ be an orientation-preserving diffeomorphism and note that by Lemma 31.1.6 the restriction $f|_{D^1_-} : D^1 \to S^1$ to the bottom half of the circle is isotopic to the identity. By the isotopy extension theorem, we may thus assume that $f|_{D^1_-} = \text{id}$ and hence it is uniquely determined by the differential $f|_{D^1_+} : D^1 \to D^1$. This is the identity on the boundary and is isotopic to the identity by linear interpolation. □

Theorem 31.2.2. *Every connected compact orientable surface (without boundary) is diffeomorphic to Σ_g for some $g \geqslant 0$.*

Proof. Any such surface Σ admits a handle decomposition. By handle rearrangement we may assume that it is given by taking k 0-handles, attaching m 1-handles, and then attaching n 2-handles to the remaining boundary circles.

Since attaching an n-handles cannot two components, it must be the case that the union of the 0- and 1-handles is connected. If $k \geqslant 2$ there must hence be a 1-handle connecting two different 0-handles, and we can remove a pair of a 0- and 1-handle using handle cancellation. Thus we may assume that $k = 1$, and by applying the same argument to reversed handle decomposition we may also assume that $n = 1$.

Each 1-handle is attached to the 0-handle (ϕ_0) along an embedding $\partial D^1 \times \{0\} \to \partial D^2$, and these must be orientation-preserving or orientation-reversing (with our outward-pointing convention for boundary orientations) or the surface would not be orientable. Pick a first 1-handle (ϕ_1) and note that there must be a second 1-handle (ϕ_2) which connects the two regions between its attaching strips, as otherwise a single 2-handle could not close the surface. This implies the boundary of $(\phi_0) + (\phi_1) + (\phi_2)$ is connected and hence we can slide the attached strips for the remaining 1-handles to clear the indicated regions;

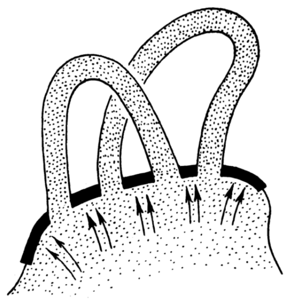


Figure 31.2 The emptied attaching regions (from [FM97]).

Repeating this argument we get g pairs of alternating 1-handles, and thus a diffeomorphism to Σ_g. □

31.2.2 The non-orientable case

If we have two k-dimensional manifolds M and N with embeddings $D^k \hookrightarrow M$ and $D^k \hookrightarrow N$, we can construct a new k-dimensional manifold

$$M \# N := (M \backslash \mathrm{int}(D^k)) \cup_{\partial D^k} (N \backslash \mathrm{int}(D^k))$$

called the *connected sum*. Using Lemma 31.1.6 and the isotopy extension theorem, $M \# N$ is in fact independent of the choice of embeddings as long as M and N are connected and either (i) oriented and we use orientation-preserving embeddings, (ii) non-orientable. For example, we have

$$\Sigma_g \cong \#_g \Sigma_1,$$

and it is possible to similarly define a non-orientable surface

$$\#_h \mathbb{R}P^2.$$

For example, the Klein bottle is diffeomorphic to $\#_2 \mathbb{R}P^2$.

Theorem 31.2.3. *Every connected compact non-orientable surface (without boundary) is diffeomorphic to $\#_h\mathbb{R}P^2$ for some $h \geqslant 1$.*

Proof sketch. We argue as in the orientable case to see that Σ has a handle decomposition with a single 0-handle, h 1-handles, and a single 2-handle. For it be non-orientable, it must be the case that at least 1-handle is twisted—the attaching map is orientation-preserving on one strip and orientation-reversing on the other—and given one such twisted strip we can clear an indicated region: and

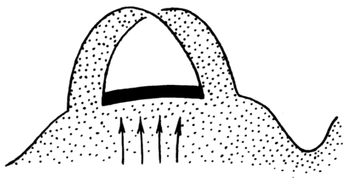


Figure 31.3 An emptied attaching region (from [FM97]).

continuing with the remaining 1-handles we can by induction either make them a collection of twisted strips or a collection of alternating untwisted 1-handles. The latter can be transformed into a collection of twisted strips: □

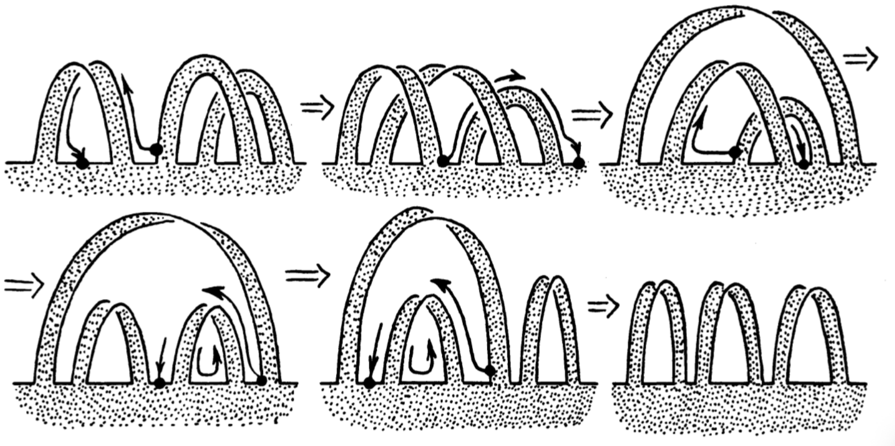


Figure 31.4 Making alternating untwisted strips into twisted strips (from [FM97]).

Chapter 32

Exotic 7-spheres I

Our final goal is to construct of a smooth manifold which is homeomorphic to S^7 but not diffeomorphic to it, an *exotic sphere*. Today prove the first statement, and the latter we will outline in the next lecture.

32.1 Reeb's theorem

If M is compact, then any smooth $f\colon M \to \mathbb{R}$ has to have a minimum and a maximum. Thus any Morse function on M has at least two critical points. What happens if it has exactly two critical points?

Theorem 32.1.1 (Reeb)**.** *If a compact k-dimensional manifold M admits a Morse function with exactly two critical points, then M is homeomorphic to S^k.*

Proof. Let p be such that $f(p) = a$ is the minimum and q be such that $f(q) = b$ is the maximum. By the Morse lemma, we can find an $\epsilon > 0$ small enough so that the following is true: $M_{\leqslant a+\epsilon}$ is diffeomorphic (using the coordinates $x_1, \ldots, x_k$) to a little disk $D^k_\epsilon(a) = \{(x_1, \ldots, x_k) \mid \sum_{i=1}^k x_i^2 \leqslant \epsilon\}$, and similarly $M_{\geqslant b-\epsilon}$ is diffeomorphic to a little disk D^k_ϵ. Hence their boundaries $M_{a+\epsilon}$ and $M_{b+\epsilon}$ are diffeomorphic to $(k-1)$-spheres. The region $M_{[a+\epsilon,b-\epsilon]}$ contains no critical points, so is diffeomorphic to $M_{a+\epsilon} \times [0, 1]$.

Thus M is obtained by gluing a cylinder $M_{[a+\epsilon,b-\epsilon]} = S^{k-1} \times [0, 1]$ to two disks D^k given by $M_{\leqslant a+\epsilon}$ and $M_{\geqslant b-\epsilon}$. The diffeomorphism is such that

$$S^{k-1} \times \{0\} = M_{a+\epsilon} \times \{0\} \longrightarrow M_{a+\epsilon} = \partial M_{\leqslant a+\epsilon} = S^{k-1}$$

is the identity, so doing this first gluing we see that there are diffeomorphisms

$$\sigma\colon M_{\leqslant b-\epsilon} \cong D^k \cup (S^{k-1} \times [0, 1]) \cong D^k.$$

However, we have no control over the diffeomorphism

$$g\colon S^{k-1} \times \{1\} = M_{a+\epsilon} \times \{1\} \longrightarrow M_{b-\epsilon} = \partial M_{\geqslant b-\epsilon} = S^{k-1}.$$

The best we can do is the following: by Proposition 32.1.2 there exists a *homeomorphism* $G\colon D^k \to D^k$ extending this diffeomorphism. That is, we can find a homeomorphism

$$\rho\colon M_{\geqslant b-\epsilon} \longrightarrow D^k,$$

which is compatible with σ. Then we can write a homeomorphism $M \to S^k$ as follows:

$$\begin{aligned} M = M_{\leqslant b-\epsilon} \cup M_{\geqslant b-\epsilon} &\longrightarrow S^k = D^k \cup D^k \\ p &\longmapsto \begin{cases} \sigma(p) & \text{if } p \in M_{\leqslant b-\epsilon}, \\ \rho(p) & \text{if } p \in M_{\geqslant b-\epsilon}. \end{cases} \end{aligned} \tag{32.1}$$

□

Proposition 32.1.2 (Alexander trick). *Every homeomorphism (so in particular diffeomorphism) $g\colon S^{k-1} \to S^{k-1}$ extends to a homeomorphism $G\colon D^k \to D^k$.*

Proof. In radial coordinates, it is given by $G(r, \theta) := (r, g(\theta))$. □

Remark 32.1.3. For later use, we point out that if $g\colon S^{k-1} \to S^{k-1}$ extended to D^k as a *diffeomorphism*, then the formula (32.1) shows that M is diffeomorphic to S^k.

32.2 Exotic 7-spheres

We will now describe some 7-dimensional manifolds and prove that they are homeomorphic to S^7. We will in the next lecture give a brief explanation why these are not diffeomorphic to S^7, a result due to Milnor [Mil56a].

32.2.1 Milnor's construction

The unit norm quaternions $S(\mathbb{H})$ on $S(\mathbb{H})$ by multiplication on the left and the right. Thus we can write down for each pair of integers (i, j) a diffeomorphism

$$\begin{aligned} S(\mathbb{H}) \times S(\mathbb{H}) &\longrightarrow S(\mathbb{H}) \times S(\mathbb{H}) \\ (x, y) &\longmapsto (x, x^i y x^j). \end{aligned}$$

We can use this to construct 7-dimensional manifolds $X_{i,j}$ as follows: we start with two copies $D^4 \times S(\mathbb{H})$. Now we recall that $S(\mathbb{H}) \cong S^3$, so each of these has boundary $S^3 \times S(\mathbb{H}) \cong S(\mathbb{H}) \times S(\mathbb{H})$. We identify these using the above diffeomorphism. Each of these is a 3-sphere bundle over S^4.

To endow this topological space with a smooth structure, we use the existence of collars. We can avoid the use of these technical tools by gluing along open subsets in the base instead, thinking of the base as a one-point compactified $\mathbb{H}$. To do so, take two copies of $\mathbb{H} \times S(\mathbb{H})$ and identify the open subsets $(\mathbb{H}\backslash 0) \times S(\mathbb{H})$ using the diffeomorphism

$$\begin{aligned} (\mathbb{H}\backslash 0) \times S(\mathbb{H}) &\longrightarrow (\mathbb{H}\backslash 0) \times S(\mathbb{H}) \\ (x, y) &\longmapsto \left(\frac{x}{||x||^2}, \frac{x^i y x^j}{||x||^{i+j}} \right). \end{aligned}$$

Here $||x||^2 = ||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$ is the (squared) quaternion norm.

Example 32.2.1. $X_{0,0} \cong S^4 \times S^3$ and $X_{1,0} \cong S^7$.

Proposition 32.2.2. *If $i + j = 1$, $X_{i,j}$ admits a Morse function with two critical points.*

Proof. We start in the first chart $\mathbb{H} \times S(\mathbb{H})$, extending to the remaining 3-sphere $\{\infty\} \times S(\mathbb{H})$ later. The idea is to take the real part $\Re(x) = \Re(a + bi + cj + dk) = a$ on the fibers, scaled by a suitable function of norm of the base $\mathbb{H} \cup \infty$ to localize all critical points over 0:

$$f(x, y) = \frac{\Re(y)}{\sqrt{1 + ||x||^2}}.$$

For its derivative to vanish, certainly the partial derivatives of $\Re(y)$ with respect to the coordinates of y have to vanish. The function $\Re$ on $S(\mathbb{H})$ is just the height function on S^3, so this occurs only if $y = \pm 1$. A further condition is then that the partial derivatives of $1/\sqrt{1 + ||x||^2}$ with respect to the coordinates of x has to vanish, and this only happens when $x = 0$. We leave to the reader to check that the maximum at $(x, y) = (0, 1)$ and the minimum at $(x, y) = (0, -1)$ are non-degenerate.

We claim that in the other chart, the Morse function is given by

$$f(x', y') := \frac{\Re(x'(y')^{-1})}{\sqrt{1 + ||x'||^2}}.$$

Indeed, when substituting the coordinate change

$$(x', y') = \left(\frac{x}{||x||^2}, \frac{x^i y x^j}{||x||} \right),$$

we get, using cyclic invariance of $\Re$ and the fact that $\Re(y^{-1}) = \Re(y)$,

$$\begin{aligned} f(x', y') &= \frac{\Re(x'(y')^{-1})}{\sqrt{1 + ||x'||^2}} \\ &= \frac{||x||}{||x||^2} \frac{\Re(x x^{-j} y^{-1} x^{-i})}{\sqrt{1 + ||x||^2/||x||^4}} \\ &= \frac{1}{||x||} \frac{\Re(y^{-1})}{\sqrt{1 + 1/||x||^2}} \\ &= \frac{\Re(y)}{\sqrt{1 + ||x||^2}}. \end{aligned}$$

We know already know that $f(x', y')$ has no critical points unless possibly when $x' = 0$. But fixing $y' = 1$ and restricting to real $x' = a$, we get $f'(a, 0) = \frac{a}{\sqrt{1+a^2}}$ which has no critical point at $a = 0$. Hence $f(x', y')$ has no critical points. ☐

Thus Theorem 32.1.1 gives:

Corollary 32.2.3. *If $i + j = 1$, $X_{i,j}$ is homeomorphic to S^7.*

We will combine this with the following fact:

Theorem 32.2.4 (Milnor). *$X_{i,j}$ can not be diffeomorphic to S^7 unless $(i-j)^2 \equiv 1 \pmod 7$.*

Taking $i = 2$ and $j = -1$, we get $(i-j)^2 = 3^2 \equiv 2 \pmod 7$ and we have found an exotic sphere! In fact, Kervaire and Milnor proved that there 28 oriented exotic 7-spheres up to orientation-preserving diffeomorphism [KM63].

32.2.2 Exotic diffeomorphisms

Let us now return our attention to Reeb's theorem. Observe that the diffeomorphism g we obtained in its proof is orientation preserving, as it is the restriction of an obviously orientation-preserving diffeomorphism $M_a \times [0,1] \to M_{[a,b]}$.

Corollary 32.2.5. *There exist orientation-preserving diffeomorphisms of S^6 which are not isotopic to the identity.*

Proof. Suppose that in the case of $X_{2,-1}$, the orientation-preserving diffeomorphism g of S^6 obtained in Theorem 32.1.1 is isotopic to the identity, say by a family of diffeomorphisms g_t starting at the identity and ending at g. Think of S^6 as sitting inside of $\mathbb{R}^7$ via the standard embedding ι and apply the isotopy extension theorem to the family of embeddings

$$\iota \circ g_t \colon S^6 \longrightarrow \mathbb{R}^7.$$

We then obtain a family of compactly-supported diffeomorphisms φ_t of $\mathbb{R}^7$ such that $g_t = \varphi_t \circ \iota$. Since g_t maps S^6 to S^6, φ_t maps D^7 to D^7. Then $\rho := \varphi_1|_{D^7}$ is a diffeomorphism of D^7 extending g. As suggested in Remark 32.1.3, using it in the last part of the proof of Theorem 32.1.1 would prove that $X_{2,-1}$ is diffeomorphic to S^7, and we get a contradiction. Thus g was not isotopic to the identity. □

Remark 32.2.6 (The Gromoll–Meyer sphere). One of Milnor's exotic spheres—in fact, $X_{2,-1}$—can be obtained explicitly up to diffeomorphism as a quotient of a Lie group [GM74]. Let $\mathrm{Sp}(n)$ denote the group of $(n \times n)$-matrices with quaternion entries satisying $Q^\dagger Q = \mathrm{id} = QQ^\dagger$ where $Q^\dagger$ denotes the transpose conjugate of Q. There is an action of $\mathrm{Sp}(1)$ on $\mathrm{Sp}(2)$, where $q \in \mathrm{Sp}(1)$ acts on $Q \in \mathrm{Sp}(2)$ by

$$\begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} Q \begin{bmatrix} \overline{q} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then there is a diffeomorphism $X_{2,-1} \cong \mathrm{Sp}(2)/\mathrm{Sp}(1)$. You can use this to give an explicit formula for exotic diffeomorphism of S^6 [Dur01].

Chapter 33

Exotic 7-spheres II

We will now explain why some of Milnor's homotopy 7-spheres are *not* diffeomorphic to S^7. This is based on the signature theorem, which in turn relies on a computation of the rational oriented cobordism ring.

33.1 The signature theorem

33.1.1 Unoriented cobordism

Instead of trying to classify smooth manifolds up to diffeomorphism, one may first try to classify them up to the following weaker equivalence relation:

Definition 33.1.1. Two compact k-dimensional smooth manifolds M_0 and M_1 with empty boundary are said to be *cobordant* if there is a compact $(k+1)$-dimensional smooth manifold W such that $\partial W = M_1 \sqcup M_0$.

We call W a cobordism from M_0 to M_1. Here the "equation" $\partial W = M_1 \sqcup M_0$ means that the boundary of W comes with a diffeomorphism to the disjoint of M_0 and M_1. In particular, if M_1 is diffeomorphic to M_0 we can interpret the cylinder $M_0 \times [0,1]$ as a cobordism from M_0 to M_1.

Example 33.1.2. If $W \to \mathbb{R}$ is a proper smooth map without critical values then the Ehresmann fibration theorem says $W|_{[a,b]}$ is a cylinder between the fibres $W|_a$ and $W|_b$. The pre-image theorem says $W \to \mathbb{R}$ is just a proper smooth map with regular values $a, b \in \mathbb{R}$, then $W|_{[a,b]}$ is a cobordism between $W|_a$ and $W|_b$.

Lemma 33.1.3. *Cobordism is an equivalence relation.*

Proof. To see it is reflexive, note that the cylinder $M_0 \times [0,1]$ exhibits M_0 as cobordant to M_0. For symmetry, note that W as a cobordism from M_0 to M_1 can also be interpreted as a cobordism from M_1 to M_0. Finally, for associativity, note that if W_0 is a cobordism from M_0 to M_1 and W_1 is a cobordism from M_1 to M_2, then $W_0 \cup_{M_1} W_1$ is a cobordism from M_0 to M_2. □

Definition 33.1.4. We let Ω_k^{O} denote the set of k-dimensional compact manifolds up to cobordism. We denote the cobordism class of M by $[M]$.

Lemma 33.1.5. *Disjoint union makes Ω_k^{O} into an abelian group:*

$$[M] + [N] := [M \sqcup N].$$

Proof. It is straightforward to show that $\sqcup$ is compatible with the equivalence relation of cobordism, and gives an associative and commutative binary operation on Ω_k^{O} with identity given by $\varnothing$. It remains to see why there are inverses. To do so, we interpret $M \times [0,1]$ not as a cobordism from M to M but as a cobordism from $M \sqcup M$ to $\varnothing$, so $[M] + [M] = 0$ and thus $[M]$ is its own inverse. □

It is a consequence of the proof of this lemma that Ω_k^{O} is a 2-torsion abelian group.

Example 33.1.6 (Ω_0^{O}). A compact d-dimensional manifold M represents the identity in Ω_k^{O} if and only if it bounds a compact manifold. By the classification of 0-dimensional compact manifolds, these are given by a finite disjoint union of points. By the classification of 1-dimensional compact manifolds, a finite disjoint union of points is a boundary if and only if it consists of an even number of points. We conclude that the homomorphism

$$\begin{aligned} \Omega_0^{\mathrm{O}} &\longrightarrow \mathbb{Z}/2 \\ \{r \text{ points}\} &\longmapsto r \pmod 2 \end{aligned}$$

is an isomorphism.

Example 33.1.7 (Ω_1^{O}). Similarly, the classification of 1-dimensional compact manifolds says that every such manifold without boundary is a finite disjoint union of circles. This is the boundary of a finite disjoint union of 2-dimensional disks, so $\Omega_1^{\mathrm{O}} = 0$.

Let us assemble all Ω_k^{O} into a single graded abelian group Ω_*^{O}. In addition to disjoint unions, we can take cartesian products. We will leave the proof of the following lemma to the reader:

Lemma 33.1.8. *Cartesian product makes Ω_*^{O} into a graded-commutative algebra:*

$$[M] \cdot [N] := [M \times N].$$

The following is a deep result of Thom [Tho54], with addendum by Dold [Dol56]; its proof uses a lot of algebraic topology.

Theorem 33.1.9 (Thom, Dold)**.** *There is an isomorphism of graded-commutative algebras*

$$\Omega_*^{\mathrm{O}} \cong \mathbb{F}_2[x_i \mid i > 0 \text{ and } i \neq 2^k - 1],$$

where x_i in degree i is represented by the Dold manifold of dimension i.

This should be surprising, as it is a complete classification of smooth manifolds up to an equivalence relation that does not seem very weak. It is also quite useful, as invariants obtained by taking inverse images of regular values are often only well-defined up to cobordism and hence take values in Ω_*^{O}.

33.1.2 Oriented cobordism

As the terminology suggests, we want to modify unoriented cobordism to take into account orientations.

Definition 33.1.10. Two compact oriented k-dimensional smooth manifolds M_0, M_1 with empty boundary are said to be *oriented cobordant* if there is a compact oriented $(k+1)$-dimensional smooth manifold W such that $\partial W = M_1 \sqcup -M_0$ (recall that $-M_0$ denotes M_0 with opposite orientation).

The following is proven for oriented cobordism by taking into account orientations in the proofs for unoriented cobordism:

Lemma 33.1.11. *Oriented cobordism is an equivalence relation.*

Definition 33.1.12. We let Ω_k^{SO} denote the set of k-dimensional compact oriented manifolds up to oriented cobordism. We denote the cobordism class of M by $[M]$.

Lemma 33.1.13. *Disjoint union makes Ω_k^{O} into an abelian group, and cartesian product makes the graded abelian group Ω_*^{SO} into a graded-commutative algebra.*

If you go through the proof of this lemma, you will learn that the inverse of $[M]$ is $[-M]$, i.e. M with the opposite orientation. In particular, it is *not* the case that Ω_*^{SO} consists of 2-torsion groups. The graded-commutativity comes from the fact that $M \times N$ is orientation by appending to the orientation of $T_m M$ that of $T_n N$, so if one reverses the order the orientation changes if and only if both M and N are odd-dimensional.

Example 33.1.14 (Ω_0^{SO} and Ω_1^{SO})*.* The classification of compact oriented 0- and 1-dimensional manifolds saying that these are a finite disjoint union of oriented points or a finite disjoint union of circles. This can be used to prove that

$$\begin{aligned} \Omega_0^{\mathrm{SO}} &\longrightarrow \mathbb{Z} \\ \left\{ \begin{array}{c} r \text{ positively oriented points} \\ \text{and } s \text{ negatively oriented} \\ \text{points} \end{array} \right\} &\longmapsto r - s \end{aligned}$$

is an isomorphism, and that $\Omega_1^{\mathrm{SO}} = 0$.

Example 33.1.15 (Ω_2^{SO})*.* The classification of compact oriented surfaces says that each of these is a disjoint union of Σ_g for some $g \geqslant 0$. Each of these bounds a solid handlebody, so $\Omega_2^{\mathrm{SO}} = 0$.

The oriented cobordism ring is harder to describe, so we settle for its rationalization $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$. The following is again a deep result of Thom [Tho54]:

Theorem 33.1.16 (Thom)**.** *There is an isomorphism of graded-commutative algebras*

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[z_{4i} \mid i > 0]$$

where z_{4i} in degree $4i$ is represented by $\mathbb{C}P^{2i}$.

Example 33.1.17. That this is not the full story can be seen by the computation that $\Omega_5^{\mathrm{SO}} = \mathbb{Z}/2$, generated by $[\mathrm{SU}(3)/\mathrm{SO}(3)]$. It is known that all torsion is 2-torsion, and $\Omega_*^{\mathrm{SO}}/\mathrm{tors}$ is a free polynomial ring generated by the Milnor manifolds.

It is outside the scope of this course, but for each oriented manifold there are invariants

$$p_i(TM) \in H^{4i}(M) \qquad \text{for } i \geqslant 0,$$

called *Pontryagin classes*. As the notation suggests, these makes sense for any oriented vector bundle and here we are just applying them to TM. They record to what extent the tangent bundle is a non-trivial vector bundle. For example, if the tangent bundle M is trivial, e.g. because it is a Lie group, they all vanish.

For a compact oriented $4k$-dimensional manifold with empty boundary, one can extract from these cohomology classes a number as follows: if $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_s$ is a consequence of positive integers (possibly repeated) such that $i_1 + \cdots + i_s = k$, we take

$$\int_M p_{i_1}(TM) \cdots p_{i_s}(TM) \in \mathbb{R}.$$

It is a non-trivial fact that these numbers are in fact integers, and give homomorphisms

$$\int_M p_I \colon \Omega_{4k}^{\mathrm{SO}} \longrightarrow \mathbb{Z} \qquad \text{for } I = (i_1, \ldots, i_s) \text{ with sum } k$$

called *Pontryagin numbers*.

Tensoring with the rationals, we get linear maps $\Omega_{4k}^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$. Thom proved that these are linearly independent. As the number of sequences I is the same as dimension of $\mathbb{Q}[z_{4i} \mid i > 0]$ in degree $4k$, equal to the number of partitions $p(r)$ of r, we get:

Proposition 33.1.18 (Thom)**.** *The linear map $\bigoplus_I \int_M p_I \colon \Omega_{4k}^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}^{p(r)}$ is an isomorphism.*

Example 33.1.19. If $\int_M p_I(M) = 0$ for all sequences I, then there exists some $N \geqslant 1$ such that $\bigsqcup_N M$ bounds a compact oriented manifold.

33.1.3 The signature

Suppose that M is a compact oriented even-dimensional manifold, say of dimension $k = 2r$. Then there is a bilinear form

$$\begin{aligned} \langle -, - \rangle \colon H^r(M) \otimes H^r(M) &\longrightarrow \mathbb{R} \\ [\omega] \otimes [\nu] &\longmapsto \int_M \omega \wedge \nu. \end{aligned}$$

By graded-commutativity of the wedge product, this is anti-symmetric if r is odd and symmetric if r is even. By Poincaré duality it is non-degenerate.

For r odd, by Problem 63 there exists a *sympletic* basis $e_1, \ldots, e_s, f_1, \ldots, f_s$ of $H^r(M)$. This means that it satisfies

$$\langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle, \qquad \text{and} \qquad \langle e_i, f_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

That is, in this basis it is given by the skew-symmetric matrix

$$\begin{bmatrix} 0 & \mathrm{id}_s \\ -\mathrm{id}_s & 0 \end{bmatrix}.$$

In particular, we can not obtain any information from it that the Betti number $\beta_i(M) := \dim H^r(M)$ does not already tell us.

For r even, we can use Sylvester's theorem—a direct consequence of the spectral theorem for symmetric matrices—which says that there exists a basis $e_1, \ldots, e_s, f_1, \ldots, f_t$ of $H^r(M)$ such that

$$\langle e_i, f_j \rangle = 0, \qquad \langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \qquad \langle f_i, f_j \rangle = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

That is, in this basis it is given by the symmetric matrix

$$\begin{bmatrix} \mathrm{id}_s & 0 \\ 0 & -\mathrm{id}_t \end{bmatrix}.$$

The numbers s and t are unique, and from them we extract the following invariant:

Definition 33.1.20. If M is a compact oriented $4r$-dimensional manifold, then its *signature* $\sigma(M)$ is given by $s - t$.

By construction, the signature is additive in disjoint unions and reserving the orientation multiplies it by -1. Using the following example, any integer can be realized as the signature of a $4r$-dimensional manifold.

Example 33.1.21. The signature of $\mathbb{C}P^{2i}$ is 1.

Example 33.1.22. The signature of the $K3$-manifold is -16.

The signature is a cobordism-invariant

We will now prove that the signature only depends on the oriented cobordism class of M. To do so, it suffices to prove that if a $4r$-dimensional compact oriented manifold M bounds a $(4r+1)$-dimensional compact oriented manifold W then $\sigma(M) = 0$. Indeed, if M_0 is oriented cobordant to M_1, then this implies $\sigma(M_0 \sqcup -M_1) = 0$ or equivalently $\sigma(M_0) - \sigma(M_1) = 0$.

Lemma 33.1.23. *Let $i\colon M \hookrightarrow W$ denote the inclusion and take $[\omega] \in H^{4r}(W)$. Then $\int_M i^*\omega = 0$.*

Proof. By Stokes' theorem we have $\int_M i^*\omega = \int_N d\omega = 0$ because ω is closed. $\square$

We will use the following algebraic observation.

Lemma 33.1.24. *Suppose we have a $\mathbb{R}$-vector space V of dimension $2n$ with non-degenerate symmetric bilinear form $\langle -,-\rangle \colon V \otimes V \to \mathbb{R}$ which has an n-dimensional subspace $W \subset V$ such that the restriction $\langle -,-\rangle|_W \colon W \otimes W \to \mathbb{R}$ is identically zero. Then we have $\sigma(V) = 0$.*

Proof. The proof is by induction over n. Fix $e \in W$, then by non-degeneracy there is an $f \in V$ such that $\langle e, f\rangle = 1$. By replacing f by $f - \frac{1}{2}\langle f, f\rangle e$ we may assume $\langle f, f\rangle = 0$. Then on the linear subspace $U = \operatorname{span}(e, f)$, the bilinear form $\langle -,-\rangle$ has signature 0, and $V = U \oplus U^{\perp}$. As $U^{\perp}$ is $2(n-1)$-dimensional, $W \cap U^{\perp}$ is $(n-1)$-dimensional, and $\langle -,-\rangle$ vanishes identically on it, we may invoke the induction hypothesis. □

Proposition 33.1.25. *If a $4r$-dimensional compact oriented manifold M bounds a $(4r+1)$-dimensional compact oriented manifold W then $\sigma(M) = 0$.*

Proof. It suffices to prove that $H^{2k}(M)$ is of dimension $2n$ and contains an n-dimensional subspace on which $\langle -,-\rangle$ vanishes identically. We claim that the image of $i^* \colon H^{2k}(W) \to H^{2k}(M)$ has the desired property. By Lemma 33.1.23 the bilinear form $\langle -,-\rangle$ vanishes on it, so it suffices to prove that its dimension is half of that $H^{2k}(M)$.

The long exact sequence of a pair and Poincaré–Lefschetz duality assemble to a commutative diagram

$$\begin{tikzcd}
H^{2k}(N) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^{2k}(M) \arrow[r] \arrow[d, "\cong"] & H^{2k+1}(N,M) \arrow[d, "\cong"] \\
H^{2k+1}(N,M)^* \arrow[r] & H^{2k}(M)^* \arrow[r, "(i^*)^*"] & H^{2k}(N)^*
\end{tikzcd}$$

Our starting point is the tautological equation:

$$\dim H^{2k}(M) = \dim \operatorname{im}(i^*) + \dim \operatorname{im}(i^*)^{\perp}.$$

On the one hand, the isomorphism of the top row to the bottom row and exactness gives

$$\dim \operatorname{im}(i^*) = \dim \ker((i^*)^*).$$

On the other hand, we have

$$\dim \operatorname{im}(i^*)^{\perp} = \dim \ker((i^*)^*).$$

because $\lambda \colon H^{2k}(M) \to \mathbb{R}$ is in the kernel of $(i^*)^*$ if and only if it annihilates the image of i^*. We thus get $\dim H^{2k}(M) = 2 \dim \operatorname{im}(i^*)$ and the result follows. □

The signature theorem

What we have just proved implies that the signature gives a surjective homomorphism

$$\sigma \colon \Omega_{4k}^{\mathrm{SO}} \longrightarrow \mathbb{Z},$$

which upon rationalization gives a linear functional $\sigma\colon \Omega_{4k}^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$.

By Proposition 33.1.18 this is a linear combination with rational coefficients of Pontryagin numbers. Hirzebruch determined what these coefficients are in terms of the coefficients of the Taylor series expansion of $\frac{\sqrt{z}}{\tanh(\sqrt{z})}$ around z. We shall not describe this procedure, but will remark that is easily implemented on a computer.

Theorem 33.1.26 (Hirzebruch)**.** *The signature of a $4k$-dimensional compact oriented manifold is given by*

$$\sigma(M) = \int_M L_k(p_1(TM), \ldots, p_k(TM))$$

where

$$\begin{aligned}
L_0 &= 1 \\
L_1 &= \tfrac{1}{3}p_1 \\
L_2 &= \tfrac{1}{45}(7p_2 - p_1^2) \\
L_3 &= \tfrac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3) \\
L_4 &= \tfrac{1}{14175}(381p_4 - 71p_1p_3 - 19p_2^2 + 22p_1^2p_2 - 3p_1^4) \\
&\ \ \textit{etc.}
\end{aligned}$$

This is a quite remarkable theorem. A priori, all we know about the Pontryagin numbers is that they are integers. However, as the signature is by definition an integer, the signature theorem imposes intricate arithmetic conditions on these numbers.

33.2 Application to exotic 7-spheres

We will known explain why some of Milnor's manifolds $X_{i,j}$ are not diffeomorphic to S^7, though do not have the tools to fill in the details proof, which would require at least a course in algebraic topology. The idea is straightforward, however: $X_{i,j}$ bounds a 4-disk bundle $W_{i,j}$ over S^4 and if it were diffeomorphic to S^7 then we can glue a D^8 along it to get a compact oriented manifold which contradicts the signature theorem, unless the condition in the theorem is satisfied.

Theorem 33.2.1 (Milnor)**.** *$X_{i,j}$ can not be diffeomorphic to S^7 unless* $(i-j)^2 \equiv 1 \pmod 7$.

Proof sketch. The $X_{i,j}$, given by 3-sphere bundles over S^4, naturally bound an 8-dimensional manifold $W_{i,j}$; the corresponding 4-disk bundle over S^4.

Associated to any oriented compact 7-dimensional M which bounds a compact oriented 8-dimensional manifold W, there are three invariants $\sigma(W, \partial W)$, $\int_{W,\partial W} p_1^2$, and $\int_{W,\partial W} p_2$. We will not define these, but they are relative versions of the signature and Pontryagin numbers which we discussed before, and in particular are all integers.

If we have two such W's, say W_1 and W_2, we can form the closed oriented manifold $V := W_1 \cup_M W_2$. Its invariants are related to the relative ones by the equations

$$\begin{aligned}
\sigma(V) &= \sigma(W_1, \partial W_1) - \sigma(W_2, \partial W_2),\\
\int_V p_1^2(TV) &= \int_{W_1,\partial W_1} p_1^2(TW_1) - \int_{W_2,\partial W_2} p_1^2(TW_2),\\
\int_V p_2(TV) &= \int_{W_1,\partial W_1} p_2(TW_1) - \int_{W_1,\partial W_1} p_2(TW_2).
\end{aligned}$$

The Hirzebruch signature theorem tells for closed V

$$45\,\sigma(V) = 7\int_V p_2(TV) - \int_V p_1^2(TV).$$

Thus we see that

$$\lambda(M) := 45\,\sigma(W, \partial W) - \int_{W,\partial W} p_1^2(TW) \pmod 7 \in \mathbb{Z}/7$$

is independent of W. It is an invariant of M.

Let us return to the task at hand. On the one hand, one may use the construction of $W_{i,j}$ to compute

$$\sigma(W_{i,j}, \partial W_{i,j}) = 1 \qquad \text{and} \qquad \int_{W_{i,j},\partial W_{i,j}} p_1^2(TW_{i,j}) = 4(i-j)^2.$$

Since $45 \equiv 3 \pmod 7$ and $4^{-1} = 2 \pmod 7$, we get $\lambda(W_{i,j}) = (i-j)^2 - 1$.

On the other hand, if $W_{i,j}$ is diffeomorphic to S^7 it bounds D^8 and one may use this to compute

$$\sigma(D^8, \partial D^8) = 0 \qquad \text{and} \qquad \int_{D^8,\partial D^8} p_1^2(TD^8) = 0.$$

Since $\lambda(X_{i,j})$ is independent of the bounding manifold, this implies that $\lambda(W_{i,j}) = 0$. Comparing these values we see that a necessary condition for $W_{i,j}$ to be diffeomorphic to S^7 is that $(i-j)^2 \equiv 1 \pmod 7$. $\square$

33.3 Problems

Problem 62 (Cobordism is an algebra)**.** Prove Lemma 33.1.8.

Problem 63 (Symplectic bases)**.** Prove that V is a finite-dimensional $\mathbb{R}$-vector space with non-degenerate anti-symmetric bilinear form $\langle -, - \rangle \colon V \otimes V \to \mathbb{R}$, then it admits a symplectic basis.

Problem 64 (Signature is multiplicative)**.** Use the Künneth theorem of Problem 60 to prove that

$$\sigma(M \times N) = \sigma(M)\sigma(N).$$

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